

High order three-term recursions, Riemann-Hilbert minors and Nikishin systems on star-like sets

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Abstract

We study monic polynomials $Q_n(x)$ generated by a high order three-term recursion $xQ_n(x) = Q_{n+1}(x) + a_{n-p}Q_{n-p}(x)$ with arbitrary $p \geq 1$ and $a_n > 0$ for all n . The recursion is encoded by a two-diagonal Hessenberg operator H . One of our main results is that, for periodic coefficients a_n and under certain conditions, the Q_n are multiple orthogonal polynomials with respect to a Nikishin system of orthogonality measures supported on star-like sets in the complex plane. This improves a recent result of Aptekarev-Kalyagin-Saff where a formal connection with Nikishin systems was obtained in the case when $\sum_{n=0}^{\infty} |a_n - a| < \infty$ for some $a > 0$.

An important tool in this paper is the study of ‘Riemann-Hilbert minors’, or equivalently, the ‘generalized eigenvalues’ of the Hessenberg matrix H . We prove interlacing relations for the generalized eigenvalues by using totally positive matrices. In the case of asymptotically periodic coefficients a_n , we find weak and ratio asymptotics for the Riemann-Hilbert minors and we obtain a connection with a vector equilibrium problem. We anticipate that in the future, the study of Riemann-Hilbert minors may prove useful for more general classes of multiple orthogonal polynomials.

Keywords: Multiple orthogonal polynomial, Nikishin system, banded Hessenberg matrix, block Toeplitz matrix, Riemann-Hilbert matrix, generalized Poincaré theorem, ratio asymptotics, vector equilibrium problem, interlacing, totally positive matrix.

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1 Introduction

Let $(Q_n)_{n=0}^\infty$ be the sequence of monic polynomials generated by the recurrence relation

$$xQ_n(x) = Q_{n+1}(x) + a_{n-p}Q_{n-p}(x), \quad n \geq 0, \quad (1.1)$$

for a fixed integer $p \in \mathbb{N} := \{1, 2, 3, \dots\}$, with initial conditions

$$Q_0(x) \equiv 1, \quad Q_{-1}(x) \equiv \dots \equiv Q_{-p}(x) \equiv 0. \quad (1.2)$$

The recurrence coefficients a_n are assumed to be positive real numbers:

$$a_n > 0, \quad n \geq 0. \quad (1.3)$$

Note that for $p = 1$, (1.1) reduces to the standard three-term recurrence relation for orthogonal polynomials on the real line, in the special case of an even orthogonality measure. We will be interested in the case where $p \geq 2$, which we refer to as a *high order three-term recurrence* [1].

The assumption (1.3) implies that the zeros of Q_n are located on the *star* $S_+ := \{x \in \mathbb{C} \mid x^{p+1} \in \mathbb{R}_+\}$, and that they satisfy certain *interlacing relations*. This was demonstrated by Eiermann-Varga [12] and Romdhane [24]; see also Fig. 1 and 2 below for the case $p = 2$. In the present paper we will obtain more general interlacing relations, in the context of so-called Riemann-Hilbert minors.

The polynomials Q_n are studied in the literature under various assumptions on the recurrence coefficients a_n . He and Saff [16] show that the Faber polynomials associated with the closed domain bounded by a $(p+1)$ -cusped hypocycloid satisfy the recursion (1.1) with constant coefficients $a_n = a = 1/p$. Many properties of these Faber polynomials are obtained in [12, 16].

More properties and applications for the polynomials Q_n are obtained by Ben Cheikh-Douak [3], Douak-Maroni [9], Maroni [20] and others [22, 24]. The polynomials Q_n are often called *d-symmetric d-orthogonal polynomials* in these references (with $d := p$). An application from the normal matrix model is given in [5].

General considerations [10, 17] show that the polynomials Q_n satisfy formal *multiple orthogonality relations* with respect to certain linear functionals. Aptekarev, Kalyagin and Van Iseghem [2] obtain a stronger version of this result:

Theorem 1.1. (See [1, Th. 1.1], [2, Cor. 2]:) Suppose that $a_n > 0$ for all n and the numbers a_n are uniformly bounded. Then the polynomials $Q_n(x)$ are multiple orthogonal with respect to the measures ν_1, \dots, ν_p defined in (8.3) (see Section 8), in the sense that

$$\int Q_n(x) x^m d\nu_j(x) = 0, \quad (1.4)$$

for any $m \in [0 : \lfloor \frac{n-j}{p} \rfloor]$ and $j \in [1 : p]$.

Here $x \mapsto \lfloor x \rfloor$ denotes the ‘floor’ function and we abbreviate $[i : j] := \{i, i+1, \dots, j\}$. This notation will be used throughout the paper.

The measures ν_1, \dots, ν_p are supported on a compact subset of the star S_+ . We will call them the *orthogonality measures*. Aptekarev, Kalyagin and Saff [1] study these measures in the case where $\sum_{n=0}^{\infty} |a_n - a| < \infty$ for some $a > 0$. They obtain a formal link with Nikishin systems. In the present paper we will extend this link to the case of periodic a_n . In particular, we will obtain conditions guaranteeing that ν_1, \dots, ν_p form a *true*, rather than a formal, Nikishin system.

For any $j \in [1 : p]$, define the *second kind function* $\Psi_n^{(j)}$ by

$$\Psi_n^{(j)}(z) := \int \frac{Q_n(t)}{z-t} d\nu_j(t), \quad n \geq 0. \quad (1.5)$$

Define the *Riemann-Hilbert matrix* (briefly *RH matrix*) $Y_n(z)$ by

$$Y_n(z) = \begin{pmatrix} Q_n(z) & \Psi_n^{(1)}(z) & \dots & \Psi_n^{(p)}(z) \\ Q_{n-1}(z) & \Psi_{n-1}^{(1)}(z) & \dots & \Psi_{n-1}^{(p)}(z) \\ \vdots & \vdots & & \vdots \\ Q_{n-p}(z) & \Psi_{n-p}^{(1)}(z) & \dots & \Psi_{n-p}^{(p)}(z) \end{pmatrix}. \quad (1.6)$$

This definition is a variant of the one in Van Assche, Geronimo and Kuijlaars [28], see also [13]. The matrix $Y_n(z)$ satisfies a certain Riemann-Hilbert problem; but we will not need this here.

Denote the principal $(k+1) \times (k+1)$ minor of $Y_n(z)$ by

$$B_{k,n}(z) = \det \begin{pmatrix} Q_n(z) & \Psi_n^{(1)}(z) & \dots & \Psi_n^{(k)}(z) \\ \vdots & \vdots & & \vdots \\ Q_{n-k}(z) & \Psi_{n-k}^{(1)}(z) & \dots & \Psi_{n-k}^{(k)}(z) \end{pmatrix}, \quad (1.7)$$

for $k \in [0 : p]$. We call this the *kth principal Riemann-Hilbert minor* of Y_n . For $n < k$ we set $B_{k,n}(z) \equiv 1$. In this paper we will also work with the determinants of more general submatrices of (1.6), whose rows are not necessarily consecutive; see Section 3 and following.

Lemma 1.2. For any $k \in [0 : p]$, $B_{k,n}(x)$ is a polynomial of degree

$$\deg B_{k,n} \leq \frac{p-k}{p}(n-k).$$

Proof. First we prove that $B_{k,n}(x)$ is a polynomial. By the multi-linearity of the determinant,

$$B_{k,n}(z) = \int \cdots \int \det \begin{pmatrix} Q_n(z) & Q_n(y_1) & \cdots & Q_n(y_k) \\ \vdots & \vdots & & \vdots \\ Q_{n-k}(z) & Q_{n-k}(y_1) & \cdots & Q_{n-k}(y_k) \end{pmatrix} \frac{d\nu_1(y_1) \cdots d\nu_k(y_k)}{(z - y_1) \cdots (z - y_k)}.$$

The integrand is clearly a polynomial in z , hence $B_{k,n}$ is a polynomial. Finally, the claim about the degree of $B_{k,n}(z)$ will be a consequence of Prop. 2.6 and Lemma 2.5 in what follows. (This claim may be shown in a direct way as well.) \square

Note in particular that $\deg B_{p,n} = 0$, i.e., the determinant of the full RH matrix $Y_n(z)$ is a constant. Prop. 2.6 will imply that this constant is nonzero.

Define the two complementary ‘stars’

$$S_{\pm} := \{x \in \mathbb{C} \mid x^{p+1} \in \mathbb{R}_{\pm}\}. \quad (1.8)$$

In this paper we will prove that the zeros of $B_{k,n}$ (and of more general RH minors) are all located on the star S_+ if k is even and on the star S_- if k is odd. We will also obtain several kinds of interlacing relations between the zeros of the different RH minors.

The main focus of this paper is on the case where the recurrence coefficients a_n are asymptotically periodic of period $r \in \mathbb{N}$. This means that

$$\lim_{n \rightarrow \infty} a_{rn+j} =: b_j > 0, \quad j \in [0 : r-1], \quad (1.9)$$

for certain limiting values $b_0, \dots, b_{r-1} > 0$.

It turns out that in the asymptotically periodic case, the zeros of Q_n for $n \rightarrow \infty$ are attracted (in the sense of weak convergence) by a certain rotationally invariant subset Γ_0 of the star S_+ . Moreover, the zeros asymptotically distribute themselves according to a measure μ_0 on Γ_0 , which appears in the solution to a certain vector equilibrium problem. An example of the set Γ_0 is shown in the left picture of Fig. 1. Below we will also introduce a family of sets Γ_k and measures μ_k , $k \in [0 : p-1]$, which will be the limiting zero distributions of the RH minors $B_{k,n}$.

Define the matrix

$$F(z, x) := Z^{-1} + Z^p \operatorname{diag}(b_0, \dots, b_{r-1}) - xI_r, \quad (1.10)$$

and the algebraic curve

$$0 = f(z, x) := \det F(z, x), \quad (1.11)$$

where Z denotes the cyclic shift matrix

$$Z = \begin{pmatrix} 0 & z \\ I_{r-1} & 0 \end{pmatrix}, \quad (1.12)$$

and where I_k denotes the identity matrix of size k . If $r = 1$ then we put $Z = z$ and $b_0 =: b$. In that case, (1.11) reduces to the algebraic curve $z^{-1} + bz^p - x = 0$ in [1, 16]. The matrix $F(z, x)$ can be interpreted as the *symbol* of a block Toeplitz matrix. This is explained in Section 6.

The expression $f(z, x)$ can be expanded as a Laurent polynomial in z :

$$f(z, x) = (-1)^{r-1} z^{-1} + f_0(x) + f_1(x)z + \cdots + f_p(x)z^p, \quad (1.13)$$

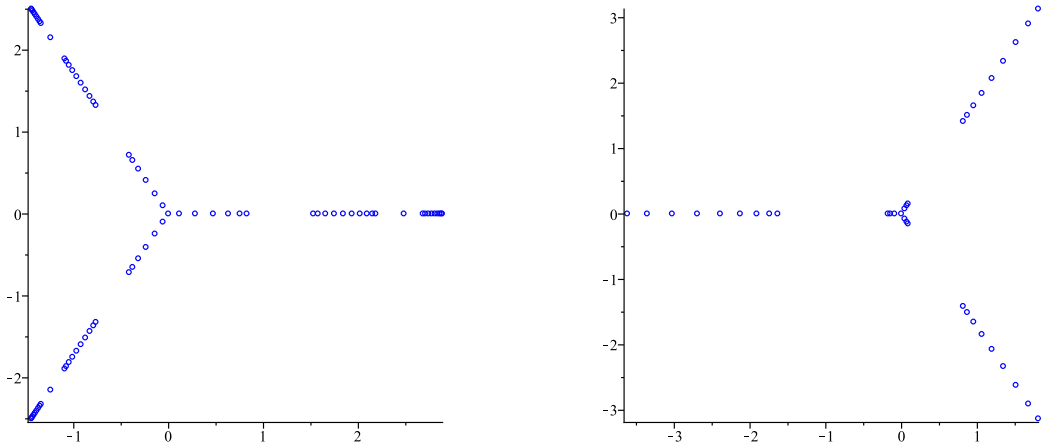


Figure 1: Zeros of Q_{80} (left) and $P_{1,80}$ (right) in the periodic case with $p = 2$ and period $r = 8$ and $(a_0, \dots, a_7) = (3, 1, 5, 2, 2, 9, 6, 1)$. The zeros of Q_n accumulate on a set $\Gamma_0 \subset S_+$ whose intersection with \mathbb{R} is $[0, 0.85] \cup [1.52, 2.19] \cup [2.67, 2.89]$ (using two digits of precision). The zeros of $P_{1,n}$ accumulate on a set $\Gamma_1 \subset S_-$ whose intersection with \mathbb{R} is $[-3.72, -1.59] \cup [-0.17, 0]$. Note that Q_{80} has an isolated zero between some of the intervals.

where each $f_k(x)$, $k \in [0 : p]$, is a polynomial in x , and

$$f_p(x) \equiv f_p = (-1)^{p(r-p)} \prod_{k=0}^{r-1} b_k. \quad (1.14)$$

The algebraic equation $f(z, x) = 0$ has precisely $p + 1$ roots $z_k = z_k(x)$, $k \in [0 : p]$ (counting multiplicities), and we order them by increasing modulus as

$$|z_0(x)| \leq |z_1(x)| \leq \dots \leq |z_p(x)| \quad (1.15)$$

for all $x \in \mathbb{C}$. If $x \in \mathbb{C}$ is such that two or more subsequent roots $z_k(x)$ in (1.15) have the same modulus then we may arbitrarily label them so that (1.15) is satisfied. It is easy to see (see e.g. [7, Sec. 4] or [29, p. 102]) that for $x \rightarrow \infty$,

$$z_0(x) = x^{-r} + O(x^{-r-1}), \quad z_k(x) = O(x^{r/p}), \quad k \in [1 : p]. \quad (1.16)$$

More precisely, for any $x \in \mathbb{C}$ there is a permutation $(\tilde{z}_k(x))_{k=1}^p$ of the set $(z_k(x))_{k=1}^p$ so that

$$\tilde{z}_k(x)^{p/d} = \left(\prod_{n=0}^{r/d-1} b_{dn+(k-1 \bmod d)} \right)^{-1} x^{r/d} (1 + o(1)), \quad k \in [1 : p], \quad (1.17)$$

as $x \rightarrow \infty$, where $d := \gcd\{p, r\}$. See [29, p. 102].

Define the sets Γ_k by

$$\Gamma_k = \{x \in \mathbb{C} \mid |z_k(x)| = |z_{k+1}(x)|\}, \quad k \in [0 : p-1]. \quad (1.18)$$

It turns out that Γ_k is a finite union of line segments on the star S_+ if k is even and S_- if k is odd: see Fig. 1 and Theorem 2.2. The next lemma shows that Γ_k is rotationally invariant.

Lemma 1.3. (*Rotational symmetry:*) With $\omega := \exp(2\pi i/(p+1))$, we have $f(z, \omega x) = \omega^r f(\omega^r z, x)$. Hence, for any $x \in \mathbb{C}$ the sets $(z_k(\omega x))_{k=0}^p$ and $(\omega^{-r} z_k(x))_{k=0}^p$ are equal up to permutation, and each set Γ_k is invariant under rotations over $2\pi/(p+1)$.

Proof. Recalling (1.10)–(1.12), it is easy to see that $D^{-1}F(z, \omega x)D = \omega F(\omega^r z, x)$ where $D := \text{diag}(1, \omega, \omega^2, \dots, \omega^{r-1})$. This implies the lemma. \square

For any $k \in [0 : p-1]$, define the measure

$$d\mu_k(\lambda) = \frac{1}{2\pi i} \frac{1}{r} \sum_{j=0}^k \left(\frac{z'_{j+}(\lambda)}{z_{j+}(\lambda)} - \frac{z'_{j-}(\lambda)}{z_{j-}(\lambda)} \right) d\lambda \quad (1.19)$$

supported on Γ_k . Here the prime denotes the derivative with respect to λ , and $d\lambda$ denotes the complex line element on each line segment of Γ_k , according to some fixed orientation of Γ_k . Moreover, $z_{j+}(\lambda)$ and $z_{j-}(\lambda)$ are the boundary values of $z_j(\lambda)$ obtained from the $+$ -side and $-$ -side respectively of Γ_k , where the $+$ -side ($-$ -side) is the side that lies on the left (right) when moving through Γ_k according to its orientation. It turns out that μ_k is a positive measure (obviously independent of the orientation given to Γ_k) with total mass [7, Sec. 4]

$$\mu_k(\Gamma_k) = \frac{p-k}{p}, \quad k \in [0 : p-1]. \quad (1.20)$$

The measures $(\mu_k)_k$ are the minimizers to an equilibrium problem that we now describe. For any measures μ, ν on \mathbb{C} define their mutual logarithmic energy as

$$I(\mu, \nu) = \iint \log \frac{1}{|x-y|} d\mu(x) d\nu(y).$$

The logarithmic energy of the measure μ is defined as $I(\mu) = I(\mu, \mu)$.

We call a vector of positive measures $\vec{\nu} = (\nu_0, \dots, \nu_{p-1})$ *admissible* if ν_k has finite logarithmic energy, ν_k is supported on Γ_k , and ν_k has total mass $\nu_k(\Gamma_k) = \frac{p-k}{p}$, $k \in [0 : p-1]$. The *energy functional* J is defined by

$$J(\vec{\nu}) = \sum_{k=0}^{p-1} I(\nu_k) - \sum_{k=0}^{p-2} I(\nu_k, \nu_{k+1}). \quad (1.21)$$

The (*vector*) *equilibrium problem* is to minimize the energy functional (1.21) over all admissible vectors of positive measures $\vec{\nu}$. The equilibrium problem has a unique solution which is given by the measures μ_k in (1.19), see [7].

2 Statement of results

2.1 Limiting zero distribution of Riemann-Hilbert minors

Denote the normalized zero counting measure of $B_{k,n}$, $k \in [0 : p-1]$, by

$$\mu_{k,n} := \frac{1}{n} \sum_{x|B_{k,n}(x)=0} \delta_x, \quad (2.1)$$

where δ_x is the Dirac measure at x and each zero is counted according to its multiplicity. Lemma 1.2 shows that $\mu_{k,n}$ has total mass at most $(p-k)/p$. Now we state our first main theorem.

Theorem 2.1. *Assume we have asymptotically periodic recurrence coefficients (1.9), and define $\mu_{k,n}$, μ_k as in (2.1) and (1.19). Then for any $k \in [0 : p - 1]$, the measures $\mu_{k,n}$ weakly converge to the measure μ_k on Γ_k as $n \rightarrow \infty$. This means that*

$$\lim_{n \rightarrow \infty} \int \phi(x) d\mu_{k,n}(x) = \int \phi(x) d\mu_k(x) \quad (2.2)$$

for any bounded continuous function ϕ .

Theorem 2.1 will be proved in Section 7 with the help of a ‘normal family’ estimate for the ratio of two RH minors (Section 5), and using the generalized Poincaré theorem. In fact, we will use a multi-column version of the generalized Poincaré theorem (Lemma 7.2). This approach yields not only weak asymptotics but also ratio asymptotics for the RH minors, as we explain in Section 7, see e.g. (7.23) or (7.28). Moreover, we will see that Theorem 2.1 remains valid with $B_{k,n}$ replaced by more general RH minors (Remark 7.7).

We point out that Theorem 2.1 for $k = 0$ could also be obtained from the normal family arguments in [4], taking into account the interlacing relations for the zeros of Q_n .

Theorem 2.1 shows that the limiting zero distribution of each Riemann-Hilbert minor $B_{k,n}$ exists and that the limiting measures are the minimizers to a vector equilibrium problem. We have reason to believe that a similar conclusion may hold for more general classes of multiple orthogonal polynomials. This may be an interesting topic for further research.

2.2 Star-like structure of Γ_k

Theorem 2.2. *Assume that (1.9) holds. Fix $k \in [0 : p - 1]$ and define Γ_k (1.18) and also*

$$\tilde{\Gamma}_k = \Gamma_k^{p+1} := \{x^{p+1} \mid x \in \Gamma_k\}. \quad (2.3)$$

Then:

- (a) $\tilde{\Gamma}_k$ is part of \mathbb{R}_+ (or \mathbb{R}_-) if k is even (or odd respectively).
- (b) $\tilde{\Gamma}_k$ is the union of n_k intervals $I_{j,k}$:

$$\tilde{\Gamma}_k = \bigcup_{j=1}^{n_k} I_{j,k}, \quad \text{with } n_k = \left\lceil \frac{k+1}{p+1} r \right\rceil - \left\lfloor \frac{kr}{p} \right\rfloor, \quad (2.4)$$

with $x \mapsto \lceil x \rceil$ and $x \mapsto \lfloor x \rfloor$ denoting the ‘ceiling’ and ‘floor’ functions. The intervals $I_{j,k}$, $j \in [1 : n_k]$ are pairwise disjoint except maybe for common endpoints.

- (c) The following conditions imply that $\tilde{\Gamma}_k$ contains 0 or ∞ :

$$\frac{k+1}{p+1} r \notin \mathbb{N} \Rightarrow 0 \in \tilde{\Gamma}_k, \quad \frac{kr}{p} \notin \mathbb{N} \cup \{0\} \Rightarrow (-1)^k \infty \in \tilde{\Gamma}_k. \quad (2.5)$$

Theorem 2.2 was formulated for the sets $\tilde{\Gamma}_k$ in (2.3). In terms of the original sets Γ_k , it implies that Γ_k lies on one of the two stars S_+ and S_- in (1.8), depending on whether k is even or odd respectively. Recall that Γ_k is rotationally invariant (Lemma 1.3).

Theorem 2.2 will be proved in Section 7.3. In the case $r = 1$ it was already obtained by Aptekarev-Kalyagin-Saff [1]; note that in that case we have $n_0 = \dots = n_{p-1} = 1$, $\tilde{\Gamma}_0 = [0, c]$ for a certain $c > 0$, and $\tilde{\Gamma}_k = (-1)^k \mathbb{R}_+$ for $k \in [1 : p - 1]$.

Remark 2.3. As mentioned in the statement of the theorem, the intervals $I_{j,k}$, $j \in [1 : n_k]$ in (2.4) are pairwise disjoint except possibly for common endpoints. We believe that such common endpoints are rare, in the sense that for a sufficiently ‘generic’ choice of the parameters $b_k > 0$, $k \in [0 : r - 1]$, all the endpoints of the intervals are distinct.

Remark 2.4. Suppose $p = 2$. Then we have two values $k = 0$ and $k = 1$, and (2.4) reduces to

$$n_0 = \left\lceil \frac{r}{3} \right\rceil, \quad n_1 = \left\lceil \frac{2r}{3} \right\rceil - \left\lfloor \frac{r}{2} \right\rfloor.$$

For example, if the period $r = 6$ then we have $n_0 = 2$ and $n_1 = 1$. Note that this is the same setting as in [19], but in the latter paper there is an additional structure on the $b_k > 0$ which implies that the two intervals of $\tilde{\Gamma}_0$ are tangent (and contain the origin), so that $\tilde{\Gamma}_0$ consists of a single contiguous interval in that case. If the period $r = 8$ then we have $n_0 = 3$ and $n_1 = 2$: see Fig. 1.

2.3 Generalized eigenvalues and interlacing

To obtain interlacing relations for the zeros of RH minors, we will use an alternative representation via generalized eigenvalue determinants that we now describe. To the recurrence (1.1) we associate the Hessenberg operator $H = (H_{i,j})_{i,j=0}^\infty$ with entries

$$\begin{cases} H_{j-1,j} &= 1, & j \geq 1, \\ H_{j+p,j} &= a_j, & j \geq 0, \\ H_{i,j} &= 0, & \text{otherwise.} \end{cases} \quad (2.6)$$

We refer to H as a *two-diagonal Hessenberg matrix*. We denote with H_n its $n \times n$ leading principal submatrix:

$$H_n = (H_{i,j})_{i,j=0}^{n-1} = \begin{pmatrix} 0 & 1 & & & 0 \\ \vdots & \ddots & \ddots & & \\ a_0 & & \ddots & \ddots & \\ & \ddots & & \ddots & 1 \\ 0 & & a_{n-p-1} & \dots & 0 \end{pmatrix}_{n \times n}. \quad (2.7)$$

The recurrence relation (1.1) can be written in matrix-vector form as

$$x(Q_0(x), Q_1(x), \dots)^T = H(Q_0(x), Q_1(x), \dots)^T, \quad (2.8)$$

where the superscript T denotes the transpose. This implies easily that $Q_n(x) = \det(xI_n - H_n)$. So the eigenvalues of H_n are the zeros of Q_n .

For $k \in [0 : p]$ we define the polynomial $P_{k,n}(x)$ as the determinant of the submatrix of $H_n - xI_n$ obtained by skipping the first k rows and the last k columns. Thus

$$P_{k,n}(x) = \det \begin{pmatrix} 0 & \dots & -x & 1 & & \\ \vdots & \ddots & & \ddots & \ddots & \\ a_0 & & \ddots & \ddots & \ddots & \\ & \ddots & & \ddots & \ddots & 1 \\ & & \ddots & & \ddots & -x \\ & & & \ddots & \ddots & \vdots \\ & & & & a_{n-p-1} & \dots & 0 \end{pmatrix}_{(n-k) \times (n-k)}. \quad (2.9)$$

The k th *generalized eigenvalues* of H_n are the numbers $x \in \mathbb{C}$ such that $P_{k,n}(x) = 0$. For $n \leq k$ we set $P_{k,n}(x) \equiv 1$. Note that for $k = p$ we have $P_{p,n}(x) \equiv a_0 \cdots a_{n-p-1} > 0$.

Lemma 2.5. *For any $k \in [0 : p]$, the polynomial $P_{k,n}(x)$ has degree*

$$\deg P_{k,n} \leq \frac{p-k}{p}(n-k).$$

Proof. This follows by a simple combinatorial argument; see e.g. [11, Proof of Prop. 2.5]. \square

Lemma 2.5 could be refined using the combinatorial formulas in Section 5. This leads to an exact formula for $\deg P_{k,n}$, depending on $n \bmod p$. We will not go into this issue here.

The fact of the matter is the following.

Proposition 2.6. *(Generalized eigenvalues versus RH minors:) For any $k \in [0 : p]$,*

$$B_{k,n}(x) = (-1)^{n(k+1) - \binom{k+1}{2}} c_k P_{k,n}(x), \quad (2.10)$$

cf. (1.7), where the constant c_k depends only on the first k moments of the measures ν_1, \dots, ν_k :

$$c_k = (-1)^k \left(\int d\nu_1(t) \right) \left(\int Q_1(t) d\nu_2(t) \right) \cdots \left(\int Q_{k-1}(t) d\nu_k(t) \right). \quad (2.11)$$

Note that in (2.10) and (2.11), we should understand $\binom{1}{2} = 0$ and $c_0 = 1$.

We point out that Prop. 2.6 remains valid for arbitrary banded Hessenberg operators, that is, for matrices $H = (H_{i,j})_{i,j=0}^\infty$ defined by

$$\begin{cases} H_{j-1,j} = 1, & j \geq 1, \\ H_{j+k,j} = a_j^{(k)}, & j \geq 0, k \in [0 : p], a_j^{(k)} \in \mathbb{C}, \\ H_{i,j} = 0, & \text{otherwise,} \end{cases} \quad (2.12)$$

so that

$$H_n = (H_{i,j})_{i,j=0}^{n-1} = \begin{pmatrix} a_0^{(0)} & 1 & & & 0 \\ \vdots & \ddots & \ddots & & \\ a_0^{(p)} & & \ddots & \ddots & \\ & \ddots & & \ddots & 1 \\ 0 & & a_{n-p-1}^{(p)} & \cdots & a_{n-1}^{(0)} \end{pmatrix}_{n \times n}. \quad (2.13)$$

We will assume that $a_j^{(p)} \neq 0$, for all j , so the entries on the p th subdiagonal of (2.13) are non-zero. We associate to H the sequence of monic polynomials $(Q_n)_{n=0}^\infty$ satisfying the $(p+2)$ -term recurrence relation (2.8), i.e.,

$$xQ_n(x) = Q_{n+1}(x) + a_n^{(0)}Q_n(x) + a_{n-1}^{(1)}Q_{n-1}(x) + \cdots + a_{n-p}^{(p)}Q_{n-p}(x), \quad n \geq 0, \quad (2.14)$$

with initial conditions

$$Q_{-1} \equiv \cdots \equiv Q_{-p} = 0, \quad Q_0 \equiv 1. \quad (2.15)$$

Prop. 2.6 will be a consequence of a result proved in Prop. 3.1 for the polynomials Q_n satisfying (2.14)–(2.15), assuming that these polynomials are *multiple orthogonal* with respect to a system of p measures, see Section 3 for more details.

Prop. 2.6 shows that RH minors can be alternatively represented as generalized eigenvalue determinants. We now state interlacing relations for the latter.

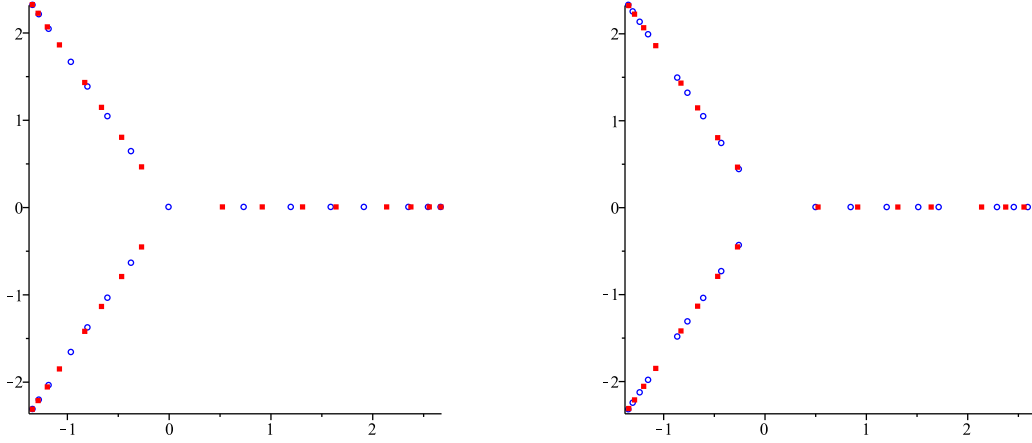


Figure 2: Left picture: zeros of Q_{23} (circles) and Q_{24} (squares). Right picture: zeros of Q_{24} (squares) and Q_{27} (circles). In these pictures we have a two-diagonal Hessenberg matrix H as in (2.6)–(2.7) with $p = 2$ and recurrence coefficients $(a_0, \dots, a_5) = (3, 2, 3, 5, 4, 1)$ extended periodically with periodicity $r = 6$.

Theorem 2.7. (*Interlacing for generalized eigenvalues:*) Let H be a two-diagonal Hessenberg matrix (2.6) with $a_j > 0$ for all j . Fix $n \in \mathbb{N}$ and $k \in [0 : p - 1]$. Then

(a) We have $P_{k,n}(x) = x^{m_{k,n}} \tilde{P}_{k,n}(x^{p+1})$, for a polynomial $\tilde{P}_{k,n}$ with $\tilde{P}_{k,n}(0) \neq 0$ and with

$$m_{k,n} = \begin{cases} (j - k)(k + 1), & \text{if } n \equiv j \pmod{p+1}, j \in [k : p], \\ (k - j)(p - k), & \text{if } n \equiv j \pmod{p+1}, j \in [-1 : k]. \end{cases} \quad (2.16)$$

The zeros of $\tilde{P}_{k,n}$ all lie in \mathbb{R}_+ (\mathbb{R}_-) if k is even (odd).

(b) Denote the zeros of $\tilde{P}_{k,n}$ and $\tilde{P}_{k,n+1}$ as $(x_i)_{i=1,2,\dots}$ and $(y_i)_{i=1,2,\dots}$ respectively, counting multiplicities and ordered by increasing modulus. We have the weak interlacing relation

$$0 < |x_1| \leq |y_1| \leq |x_2| \leq |y_2| \leq \dots$$

if $n \equiv j \pmod{p+1}$, $j \in [k : p - 1]$, and

$$0 < |y_1| \leq |x_1| \leq |y_2| \leq |x_2| \leq \dots$$

if $n \equiv j \pmod{p+1}$, $j \in [-1 : k - 1]$.

(c) Let $(x_i)_{i=1,2,\dots}$ be the zeros of $\tilde{P}_{k,n}$, as in (b), and let $(w_i)_{i=1,2,\dots}$ be the zeros of $\tilde{P}_{k,n+p+1}$, counting multiplicities and ordered by increasing modulus. We have

$$0 < |w_1| \leq |x_1| \leq |w_2| \leq |x_2| \leq \dots$$

Note that the moduli can be removed if k is even and replaced by minus signs if k is odd.

Theorem 2.7 generalizes known results for the standard eigenvalues $k = 0$ [12, 24]. The theorem will be proved in Section 4, by using the theory of totally positive matrices and extending the approach of Eiermann-Varga [12]. See also Theorems 2.12 and 4.6 below for related results.

Theorem 2.7 is illustrated in Figures 2 and 3.

Generalized eigenvalues turn out to be deeply connected to the hierarchy of functions of the (formal) *Nikishin system* generated by H . This will be the topic of Section 2.4.

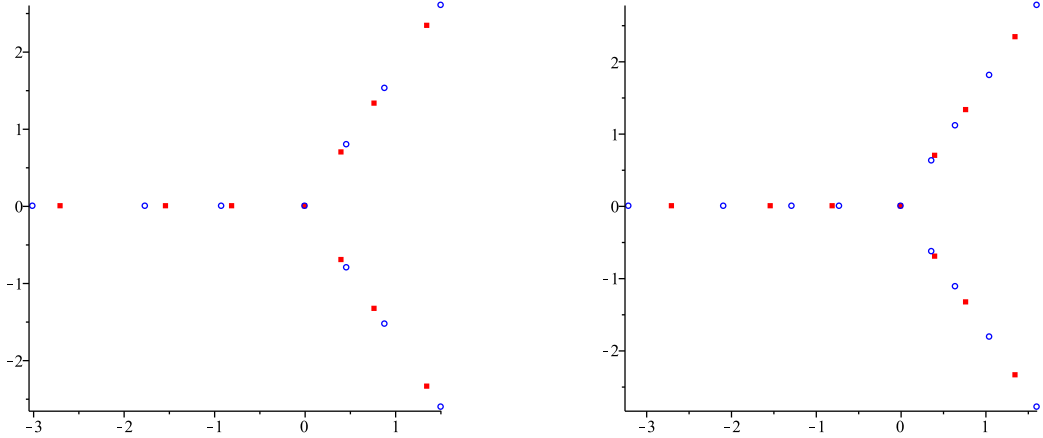


Figure 3: Left picture: zeros of $P_{1,23}$ (circles) and $P_{1,24}$ (squares). Right picture: zeros of $P_{1,24}$ (squares) and $P_{1,27}$ (circles). The matrix H is as in Figure 2.

2.4 Connection with Nikishin systems

Aptekarev-Kalyagin-Saff [1] show that, in the trace class $\sum_{k=0}^{\infty} |a_k - a| < \infty$ and with period $r = 1$, the two-diagonal operator H generates a (formal) *Nikishin system*. These objects are only formally defined however.

In this paper we will obtain a related result. It will apply to the exactly periodic case

$$a_{rn+k} = a_k = b_k, \quad n \in \mathbb{N}, \quad k \in [0 : r - 1]. \quad (2.17)$$

We assume without loss of generality that the period r is a multiple of p . We also assume that

$$\prod_{n=0}^{r/p-1} a_{pn} > \prod_{n=0}^{r/p-1} a_{pn+1} > \cdots > \prod_{n=0}^{r/p-1} a_{pn+(p-1)}. \quad (2.18)$$

Under these conditions, we will show that the polynomials Q_n are multiple orthogonal with respect to a *true* Nikishin system generated by rotationally invariant measures on the stars S_+ and S_- , coming from measures on \mathbb{R}_+ or \mathbb{R}_- with *constant sign*. There can also be possible point masses at each level of the Nikishin hierarchy.

Nikishin systems formed by measures supported on the real line were introduced by E.M. Nikishin in [23]. The same definition can be easily adapted to our context of star-like sets, as we now explain. Compare this definition with the one given in [1, Section 8.1].

Definition 2.8. Let ν_1, \dots, ν_p be a collection of p complex measures supported on the set $\Gamma_0 \cup \mathcal{A}_0$, where $\mathcal{A}_0 \subset S_+ \setminus \Gamma_0$ is a discrete set. We say that (ν_1, \dots, ν_p) forms a Nikishin system on $(\Gamma_0, \dots, \Gamma_{p-1})$ if for each $k \in [0 : p - 1]$, there exists a collection of complex measures $(\nu_{l,k})_{l=k+1}^p$ supported on $\Gamma_k \cup \mathcal{A}_k$, where \mathcal{A}_k is a discrete subset of $S_+ \setminus \Gamma_k$ (if k is even) or $S_- \setminus \Gamma_k$ (if k is odd), with the following properties:

(a) $(\nu_1, \dots, \nu_p) = (\nu_{1,0}, \dots, \nu_{p,0})$.

(b) If $d\nu_{l,k}(x) = g_{l,k}(x)dx + d\nu_{l,k}^{(s)}(x)$, $d\nu_{l,k}^{(s)}(x) \perp g_{l,k}(x)dx$, denotes the Lebesgue decomposition of $\nu_{l,k}$, $l \in [k + 1 : p]$, then

$$\frac{g_{l,k}(x)}{g_{k+1,k}(x)} = \int \frac{d\nu_{l,k+1}(t)}{x - t}, \quad x \in \Gamma_k, \quad l \in [k + 2 : p]. \quad (2.19)$$

- (c) For every $l \in [k+1 : p]$, there exists a real measure $\tilde{\nu}_{l,k}$ with constant sign (either positive or negative), supported on \mathbb{R}_+ (\mathbb{R}_-) if k is even (odd), such that

$$d\nu_{l,k}(t) = t^{k+1-l} d\tilde{\nu}_{l,k}(t^{p+1}). \quad (2.20)$$

Remark 2.9. We observe that Nikishin systems possess a hierarchical structure, with the measures (ν_1, \dots, ν_p) forming level 0 of the hierarchy. The measure $\nu_{l,k}$ is said to be at the k th level of the Nikishin hierarchy. Note that (2.20) implies that for any $k \in [0 : p-1]$, the measure $\nu_{k+1,k}$ is rotationally invariant, and the induced measure $\tilde{\nu}_{k+1,k}$ is real with constant sign. The measures $\nu_{k+1,k}$ are usually referred to as the *generating measures* of the Nikishin system. We are implicitly requiring in (2.19) that $g_{k+1,k}(x) \neq 0$ for all but finitely many $x \in \Gamma_k$.

Our main result is the following.

Theorem 2.10. *Let H be a two-diagonal Hessenberg matrix (2.6) with exactly periodic coefficients $a_j > 0$ satisfying (2.17)–(2.18), where the period r is a multiple of p . Then the orthogonality measures (ν_1, \dots, ν_p) in Theorem 1.1 form a Nikishin system on $(\Gamma_0, \dots, \Gamma_{p-1})$ (Def. 2.8). Moreover, the star-like sets $(\Gamma_k)_{k=0}^{p-1}$ are compact and the discrete sets $(\mathcal{A}_k)_{k=0}^{p-1}$ are finite.*

Theorem 2.10 will be proved in Section 8.

Remark 2.11. Theorem 2.10 was stated under the condition (2.18). In general, consider the set

$$\left\{ \prod_{n=0}^{r/p-1} a_{pn}, \prod_{n=0}^{r/p-1} a_{pn+1}, \dots, \prod_{n=0}^{r/p-1} a_{pn+(p-1)} \right\}. \quad (2.21)$$

Eq. (1.17) (with $d = p$) shows that there exists a permutation Π of $[1 : p]$ so that

$$z_{\Pi(k)}(x) = \left(\prod_{n=0}^{r/p-1} a_{pn+k-1} \right)^{-1} x^{r/p} (1 + o(1)), \quad k \in [1 : p], \quad (2.22)$$

for $x \rightarrow \infty$. This can also be seen from the derivation of (8.26) in Section 8. As a consequence, if the p numbers in (2.21) are pairwise distinct then all the Γ_k are bounded. The converse of the last statement is also true, due to [26, Lemma 3.3]. Now if the numbers (2.21) are pairwise distinct but ordered in a different way than (2.18), then a variant to Theorem 2.10 holds. We then have an additional constant or monomial term in the right hand side of (2.19). This is due to the fact that the constant α in Eq. (8.19) in Section 8 can be nonzero in this case.

We see that the key to obtaining a *true* (rather than a formal) Nikishin system is to show that the ratio between the densities (2.19) of the measures at the different levels of the Nikishin hierarchy are Cauchy transforms of measures on S_+ or S_- , associated to real measures with constant sign on \mathbb{R}_+ or \mathbb{R}_- . We will establish this requirement via a surprising connection with RH minors. In particular we will use the interlacing relations between the zeros of RH minors.

Recall the generalized eigenvalue determinant $P_{k,n}(x)$ from (2.9). We need a more general definition. For any $1 \leq k \leq l \leq p$ we define $P_{k,l,n}(x)$ as the determinant of the submatrix obtained by skipping rows $0, 1, \dots, k-1$ and columns $n-l, n-k+1, n-k+2, \dots, n-1$ of $H_n - xI_n$. If $l = k$ then we retrieve our previous definition: $P_{k,k,n}(x) \equiv P_{k,n}(x)$.

In the proof of Theorem 2.10 we need the following result on the polynomials $P_{k,l,n}(x)$.

Theorem 2.12. *(Interlacing for $P_{k,l,n}$ and $P_{k,n}$.) Let H be a two-diagonal Hessenberg matrix (2.6) with $a_j > 0$ for all j . Fix $n \in \mathbb{N}$ and $0 \leq k < l \leq p$. Then*

(a) We have $P_{k,l,n}(x) = x^{k-l+m_{k,n}} \tilde{P}_{k,l,n}(x^{p+1})$, with $m_{k,n}$ defined in (2.16) and with $\tilde{P}_{k,l,n}$ a polynomial whose zeros all lie in \mathbb{R}_+ (\mathbb{R}_-) if k is even (odd).

(b) Denote the zeros of $\tilde{P}_{k,n}(x)$ and $\tilde{P}_{k,l,n}(x)$ as $(x_i)_{i=1,2,\dots}$ and $(y_i)_{i=1,2,\dots}$ respectively, ordered by increasing modulus and counting multiplicities. We have the weak interlacing relation

$$0 \leq |y_1| \leq |x_1| \leq |y_2| \leq |x_2| \leq \dots$$

Theorem 2.12 is proved in Section 4. The precise way how Theorem 2.12 is used in the proof of Theorem 2.10 will be explained in Section 8.

Remark 2.13. The polynomial $\tilde{P}_{k,l,n}$ could have one, and at most one, zero at the origin. This happens precisely when $n \equiv j \pmod{p+1}$ for some $j \in [k : l-1]$.

2.5 Widom-type formula

In this section we state an exact formula for the polynomials Q_n in the exactly periodic case (2.17). In fact, we prove the formula for general banded Hessenberg matrices H of the form (2.12). We say that H is *exactly periodic with period r* if

$$a_{rn+k}^{(j)} = a_k^{(j)} = b_k^{(j)}, \quad n \in \mathbb{N}, \quad k \in [0 : r-1], \quad j \in [0 : p]. \quad (2.23)$$

Recall that we are assuming $b_k^{(p)} \neq 0$ for all k . Define the ‘block Toeplitz symbol’

$$F(z, x) = Z^{-1} + \sum_{k=0}^p Z^k \operatorname{diag}(b_0^{(k)}, \dots, b_{r-1}^{(k)}) - xI_r, \quad (2.24)$$

with Z as in (1.12). In the case of a two-diagonal Hessenberg matrix (2.6) this reduces to (1.10). Also define $f(z, x) = \det F(z, x)$, the roots $z_k(x)$ of the algebraic equation $f(z, x) = 0$ as in (1.15), and the sets Γ_k as in (1.18). Prop. 1.1 in [7] shows that Γ_k is a finite union of analytic arcs. Clearly, (1.13)–(1.14) remain valid in this setting (with b_k replaced by $b_k^{(p)}$).

Theorem 2.14. (*Widom-type formula:*) *With the above notations, let $x \in \mathbb{C}$ be such that the solutions $z_k(x)$ of the algebraic equation $0 = f(z, x) = \det F(z, x)$ are pairwise distinct. Then for each $n \in \mathbb{N} \cup \{0\}$ and for each $j \in [0 : r-1]$,*

$$Q_{rn+j}(x) := \det(xI_{rn+j} - H_{rn+j}) = \frac{(-1)^{r+j}}{f_p} \sum_{k=0}^p \frac{\det F^{r-1,j}(z_k(x), x)}{\prod_{i=0, i \neq k}^p (z_k(x) - z_i(x))} z_k(x)^{-n-1}. \quad (2.25)$$

Here f_p is defined in (1.14) (with b_k replaced by $b_k^{(p)}$), and we use the notation $F^{i,j}$ to denote the submatrix of F in (2.24) that is obtained by skipping the i th row and the j th column, $i, j \in [0 : r-1]$. Moreover, for all i, j, k , $\det F^{i,j}(z_k(x), x)$ is zero for only finitely many x .

Theorem 2.14 will be proved in Section 6, as a consequence of Widom’s determinant identity for block Toeplitz matrices [30, Section 6]. From (2.25) and (1.15)–(1.18) we also find:

Corollary 2.15. *The strong asymptotic formula*

$$\lim_{n \rightarrow \infty} Q_{rn+j}(x) z_0(x)^{n+1} = \frac{(-1)^{r+j}}{f_p} \frac{\det F^{r-1,j}(z_0(x), x)}{\prod_{i=1}^p (z_0(x) - z_i(x))}, \quad j \in [0 : r-1],$$

holds uniformly on compact subsets of $\mathbb{C} \setminus (\Gamma_0 \cup \mathcal{A})$ with \mathcal{A} a finite set.

Incidentally, Aptekarev et al. [1] obtain strong asymptotics for Q_n in the trace class $\sum_{k=0}^{\infty} |a_k - a| < \infty$ ($a > 0$) with period $r = 1$. By using Theorem 2.14, it is possible to extend these results to higher periods r . We will not go into this issue here.

2.6 Outline of the paper

The rest of this paper is organized as follows. In Section 3 we prove the connection between RH minors and generalized eigenvalue determinants and we introduce the concept of a general RH minor $B^{(n_0, n_1, \dots, n_k)}(z)$. In Section 4 we prove interlacing relations for generalized eigenvalues. Section 5 contains normal family estimates for the ratio between two RH minors. The remaining sections deal with asymptotically periodic coefficients a_n . The proof of the Widom-type formula for Q_n in the exactly periodic case is given in Section 6. In Section 7 we obtain weak and ratio asymptotics for RH minors and we prove Theorem 2.2 on the star-like structure of Γ_k . In Section 8 we prove Theorem 2.10 on the connection with Nikishin systems.

3 Riemann-Hilbert minors and generalized eigenvalues

In this section we prove Prop. 2.6 in the general context of banded Hessenberg operators $H = (H_{i,j})_{i,j=0}^\infty$ defined in (2.12).

Let $(Q_n)_{n=0}^\infty$ be the sequence of monic polynomials associated to the operator H , i.e., the sequence satisfying (2.14)–(2.15). We will assume in this section that the polynomials Q_n are multiple orthogonal with respect to a system of p complex measures ν_1, \dots, ν_p supported on a compact contour $\Sigma \subset \mathbb{C}$. This means that for every $j \in [1 : p]$,

$$\int_{\Sigma} Q_n(t) t^m d\nu_j(t) = 0, \quad m \in [0 : \left\lfloor \frac{n-j}{p} \right\rfloor]. \quad (3.1)$$

Define the second kind functions $\Psi_n^{(j)}$ as in (1.5).

For later use, we need a more general definition of generalized eigenvalues. Let $H_n = (H_{i,j})_{i,j=0}^{n-1}$. As in (2.9), we denote by $P_{k,n}(x)$ the determinant of the matrix obtained by skipping the first k rows and the last k columns of the matrix $H_n - xI_n$. Similarly we could skip any set of k different columns, not necessarily consecutive.

Let $k \in [0 : p]$ and let (n_0, n_1, \dots, n_k) be a $(k+1)$ -tuple of positive integers such that

$$0 \leq n_0 < n_1 < \dots < n_k \leq n_0 + p. \quad (3.2)$$

We define the *generalized eigenvalue determinant associated to (n_0, n_1, \dots, n_k)* as

$$P^{(n_0, n_1, \dots, n_k)}(x) := \det(H_{n_k} - xI_{n_k})^{(0, 1, \dots, k-1; n_0, n_1, \dots, n_{k-1})}. \quad (3.3)$$

That is, the polynomial $P^{(n_0, n_1, \dots, n_k)}(x)$ is the determinant of the submatrix obtained by skipping rows $0, 1, \dots, k-1$ and columns n_0, n_1, \dots, n_{k-1} of $H_{n_k} - xI_{n_k}$. The *generalized eigenvalues* associated to (n_0, n_1, \dots, n_k) are the numbers $x \in \mathbb{C}$ such that $P^{(n_0, n_1, \dots, n_k)}(x) = 0$. In the case $k = 0$ we put $n := n_0$ and we understand $P^{(n)}(x) = \det(H_n - xI_n) = (-1)^n Q_n(x)$.

By choosing (n_0, n_1, \dots, n_k) to be a sequence of consecutive numbers:

$$(n_0, n_1, \dots, n_k) = (n-k, \dots, n-1, n),$$

we retrieve our earlier definition $P^{(n-k, \dots, n-1, n)}(x) \equiv P_{k,n}(x)$. Similarly we can retrieve $P_{k,l,n}(x)$.

In this section we prove the following connection with Riemann-Hilbert minors.

Proposition 3.1. *Let $H = (H_{i,j})_{i,j=0}^\infty$ be the banded Hessenberg matrix (2.12) with $a_j^{(p)} \neq 0$ for all $j \geq 0$. Assume that the monic polynomials $(Q_n)_{n=0}^\infty$ (2.14)–(2.15) associated with H satisfy*

the multiple orthogonality relations (3.1), for some complex measures ν_1, \dots, ν_p supported on $\Sigma \subset \mathbb{C}$. Let $\Psi_n^{(j)}$ be the second kind functions (1.5). For any $k \in [0 : p]$, we have

$$c_k(-1)^{n_0+\dots+n_k} P^{(n_0, n_1, \dots, n_k)}(z) = B^{(n_0, n_1, \dots, n_k)}(z), \quad (3.4)$$

where

$$B^{(n_0, n_1, \dots, n_k)}(z) := \det \begin{pmatrix} Q_{n_k}(z) & \Psi_{n_k}^{(1)}(z) & \dots & \Psi_{n_k}^{(k)}(z) \\ \vdots & \vdots & & \vdots \\ Q_{n_1}(z) & \Psi_{n_1}^{(1)}(z) & \dots & \Psi_{n_1}^{(k)}(z) \\ Q_{n_0}(z) & \Psi_{n_0}^{(1)}(z) & \dots & \Psi_{n_0}^{(k)}(z) \end{pmatrix}_{(k+1) \times (k+1)}, \quad (3.5)$$

and where the constant c_k is given in (2.11).

The matrix in the right hand side of (3.5) is again a submatrix of the RH matrix in (1.6) (with $n = n_k$), although not necessarily a principal submatrix. This follows from (3.2).

Proof of Prop. 3.1. We prove (3.4) by verifying that both sides of the equation satisfy the same recurrence relation. Assume that $n_k \geq k + 1$. If we apply the cofactor expansion formula to $P^{(n_0, n_1, \dots, n_k)}(z)$ along the last row, we obtain

$$P^{(n_0, n_1, \dots, n_k)}(z) = \sum_{j=1}^{p+1} (-1)^{\sigma_j} (H_{n_k} - zI_{n_k})_{n_k-1, n_k-j} \tilde{P}^{(n_0, \dots, n_{k-1}, n_k-j)}(z), \quad (3.6)$$

where in the right hand side of (3.6), $A_{i,j}$ denotes the (i, j) entry of a matrix A , the function \tilde{P} is defined in the following way:

$$\tilde{P}^{(n_0, \dots, n_{k-1}, n_k-j)}(z) := \begin{cases} P^{(\tilde{n}_0, \tilde{n}_1, \dots, \tilde{n}_k)}(z) & \text{if } n_k - j \notin \{n_0, \dots, n_{k-1}\}, \\ 0 & \text{otherwise,} \end{cases}$$

where $(\tilde{n}_0, \tilde{n}_1, \dots, \tilde{n}_k)$ is obtained by ordering the entries of $(n_0, \dots, n_{k-1}, n_k - j)$ increasingly, and where σ_j is the sum of the row and column coordinates of the entry $(H_{n_k} - zI_{n_k})_{n_k-1, n_k-j}$ in the matrix $(H_{n_k} - zI_{n_k})^{(0, 1, \dots, k-1; n_0, n_1, \dots, n_{k-1})}$ (the definition of σ_j is used only when $n_k - j \notin \{n_0, \dots, n_{k-1}\}$). We also put $(H_{n_k} - zI_{n_k})_{i,j} := 0$ whenever $j < 0$.

To prove (3.6) we observe that the submatrix of $(H_{n_k} - zI_{n_k})^{(0, 1, \dots, k-1; n_0, n_1, \dots, n_{k-1})}$ obtained by skipping the row and column occupied by the entry $(H_{n_k} - zI_{n_k})_{n_k-1, n_k-j}$, takes the form

$$\begin{pmatrix} (H_{\tilde{n}_k} - zI_{\tilde{n}_k})^{(0, 1, \dots, k-1; \tilde{n}_0, \tilde{n}_1, \dots, \tilde{n}_{k-1})} & 0 \\ * & L \end{pmatrix},$$

where L is a lower triangular square matrix of size $n_k - 1 - \tilde{n}_k$ with 1's on the main diagonal. Hence the determinant of this submatrix equals $P^{(\tilde{n}_0, \tilde{n}_1, \dots, \tilde{n}_k)}(z)$, which yields (3.6).

Note that the recursion (3.6) is completely determined from its initial condition (determinant of an empty matrix)

$$P^{(0, 1, \dots, k)}(z) \equiv 1. \quad (3.7)$$

It is well-known (and easily checked) that the second kind functions $\Psi_n^{(k)}$ satisfy exactly the same recursion as the polynomials Q_n , in the sense that

$$x\Psi_n^{(k)}(z) = \Psi_{n+1}^{(k)}(z) + a_n^{(0)}\Psi_n^{(k)}(z) + a_{n-1}^{(1)}\Psi_{n-1}^{(k)}(z) + \dots + a_{n-p}^{(p)}\Psi_{n-p}^{(k)}(z), \quad n \geq k, \quad (3.8)$$

for any $k \in [1 : p]$. The recursion for the functions $\Psi_n^{(k)}(z)$ starts only from the index $n = k$. Assume that $n_k \geq k + 1$. Applying (2.14) and (3.8) (with $n := n_k - 1$) to the first row of (3.5) and using the linearity of the determinant, we deduce that

$$B^{(n_0, n_1, \dots, n_k)}(z) = \sum_{j=1}^{p+1} (-1)^{1+\tau_j} (H_{n_k} - zI_{n_k})_{n_k-1, n_k-j} \tilde{B}^{(n_0, \dots, n_{k-1}, n_k-j)}(z) \quad (3.9)$$

where

$$\tilde{B}^{(n_0, \dots, n_{k-1}, n_k-j)}(z) := \begin{cases} B^{(\tilde{n}_0, \tilde{n}_1, \dots, \tilde{n}_k)}(z) & \text{if } n_k - j \notin \{n_0, \dots, n_{k-1}\}, \\ 0 & \text{otherwise,} \end{cases}$$

and τ_j is the number of adjacent transpositions that are necessary to order $(n_0, \dots, n_{k-1}, n_k - j)$ increasingly, e.g. for $(n_0, n_1, n_2, n_3 - j) = (1, 4, 5, 3)$ we have $\tau_j = 2$.

If $(\tilde{n}_0, \tilde{n}_1, \dots, \tilde{n}_k)$ is obtained by ordering $(n_0, \dots, n_{k-1}, n_k - j)$ increasingly, then obviously

$$n_0 + n_1 + \dots + n_k - (\tilde{n}_0 + \tilde{n}_1 + \dots + \tilde{n}_k) = j. \quad (3.10)$$

From (3.6) and (3.10) we have

$$\begin{aligned} c_k(-1)^{n_0+n_1+\dots+n_k} P^{(n_0, n_1, \dots, n_k)}(z) \\ = \sum_{j=1}^{p+1} (H_{n_k} - zI_{n_k})_{n_k-1, n_k-j} (-1)^{\sigma_j} (-1)^j (-1)^{\tilde{n}_0+\tilde{n}_1+\dots+\tilde{n}_k} c_k \tilde{P}^{(n_0, n_1, \dots, n_k-j)}(z). \end{aligned} \quad (3.11)$$

We claim that

$$(-1)^{1+\tau_j} = (-1)^{\sigma_j+j}. \quad (3.12)$$

Let $j \geq 1$ and assume that $n_{k-1} < n_k - j$. Then $\tau_j = 0$ so the left-hand side of (3.12) is -1 . Now, if j is even then σ_j is odd and vice-versa. So (3.12) holds in this case. Now let j_1 be such that $(-1)^{1+\tau_{j_1}} = (-1)^{\sigma_{j_1}+j_1}$ and $n_k - j_1 = n_l + 1$ for some $l \in [0 : k-1]$. Assume further that the next value of j greater than j_1 for which $n_k - j \neq n_i$ for all i is $j = j_1 + q + 2$, $q \geq 0$. These assumptions imply that $\tau_j = \tau_{j_1} + q + 1$, $\sigma_j = \sigma_{j_1} + 1$, and therefore $(-1)^{1+\tau_{j_1}} = (-1)^{\sigma_{j_1}+j_1}$ implies that (3.12) is valid for j . This completes the justification of (3.12).

It follows from (3.9), (3.11) and (3.12) that for each k , the functions $B^{(n_0, n_1, \dots, n_k)}$ and $c_k(-1)^{n_0+n_1+\dots+n_k} P^{(n_0, n_1, \dots, n_k)}$ satisfy the same recurrence relations. The recursion (3.9) is also completely determined from its initial condition $B^{(0,1,\dots,k)}$, which is

$$\begin{aligned} B^{(0,1,\dots,k)}(z) &= \det \begin{pmatrix} Q_k(z) & \Psi_k^{(1)}(z) & \dots & \Psi_k^{(k)}(z) \\ \vdots & \vdots & & \vdots \\ Q_0(z) & \Psi_0^{(1)}(z) & \dots & \Psi_0^{(k)}(z) \end{pmatrix} \\ &= \det \begin{pmatrix} z^k + O(z^{k-1}) & O(z^{-2}) & O(z^{-2}) & \dots & O(z^{-2}) \\ O(z^{k-1}) & O(z^{-2}) & O(z^{-2}) & \dots & C_k z^{-1} + O(z^{-2}) \\ \vdots & \vdots & \vdots & & \vdots \\ O(z^2) & O(z^{-2}) & O(z^{-2}) & \dots & O(z^{-1}) \\ O(z) & O(z^{-2}) & C_2 z^{-1} + O(z^{-2}) & \dots & O(z^{-1}) \\ O(1) & C_1 z^{-1} + O(z^{-2}) & O(z^{-1}) & \dots & O(z^{-1}) \end{pmatrix} \end{aligned}$$

with $C_j := \int Q_{j-1}(t) d\nu_j(t)$. Expanding this determinant as a signed sum over all permutations of $(0, 1, \dots, k)$, we see that all the terms in this sum are $O(z^{-1})$ except for the one that corresponds to the permutation $(0, k, \dots, 2, 1)$:

$$B^{(0,1,\dots,k)}(z) = (-1)^{\binom{k}{2}} C_1 C_2 \dots C_k + O(z^{-1}).$$

Since we already know by Lemma 1.2 that the determinant in the left hand side is a polynomial in z , the $O(z^{-1})$ term in the right hand side vanishes. The value of c_k was chosen so that

$$c_k(-1)^{0+1+\dots+k} P^{(0,1,\dots,k)}(z) = B^{(0,1,\dots,k)}(z),$$

so the two initial conditions are the same and this concludes the proof of (3.4). \square

4 Interlacing of generalized eigenvalues

In this section we prove Theorems 2.7 and 2.12 on the interlacing of generalized eigenvalues. To this end we use some results on totally positive matrices.

4.1 Generalized eigenvalues of totally positive matrices

A matrix $A \in \mathbb{C}^{n \times m}$ is called *totally positive (TP)* if the determinant of any square submatrix of A is positive, i.e.,

$$\det A(K, L) > 0, \quad (4.1)$$

for any index sets $K \subset [0 : n - 1]$, $L \subset [0 : m - 1]$ of the same cardinality $|K| = |L|$, where we write $A(K, L)$ for the submatrix of A with rows and columns indexed by K and L , respectively. We emphasize that in the submatrix $A(K, L)$ the rows and columns are positioned in the same order given in A . *Fekete's criterion* asserts that a sufficient condition for A to be TP is that (4.1) holds for all index sets K and L formed by *consecutive* indices, i.e., $K = \{r, r - 1, \dots, r - q + 1\}$ and $L = \{c, c - 1, \dots, c - q + 1\}$ with $q := |K| = |L|$ and for suitable integers r, c .

The matrix A is called *totally nonnegative (TNN)* if we have the inequality \geq in (4.1):

$$\det A(K, L) \geq 0, \quad (4.2)$$

for all index sets $K \subset [0 : n - 1]$, $L \subset [0 : m - 1]$ with $|K| = |L|$. It is well known that TP matrices are dense in the class of TNN matrices. Moreover, the class of TP (or TNN) matrices is closed under matrix multiplication.

The theory of eigenvalues for TP matrices was developed by Gantmacher-Krein [15]. They showed that the eigenvalues of an $n \times n$ TP matrix are all positive and distinct and that they strictly interlace with those of its principal $(n - 1) \times (n - 1)$ submatrix. We need the following analogue for *generalized* eigenvalues of TP matrices, which are again defined as in Section 2.3.

Proposition 4.1. (*Generalized eigenvalues of TP matrices:*) Fix $0 \leq k < n$ and let $M \in \mathbb{C}^{(n+k) \times (n+k)}$ be a TP matrix. Then the k th generalized eigenvalues of M are simple, lie in $(0, \infty)$ if k is even and lie in $(-\infty, 0)$ if k is odd. The number of k th generalized eigenvalues of M is $n - k$. Moreover, the k th generalized eigenvalues of M and its principal leading submatrix $Q \in \mathbb{C}^{(n+k-1) \times (n+k-1)}$ are strictly interlacing.

Proof. Let N be the submatrix of M obtained after skipping the first k rows and the last k columns of M . Thus N is of size $n \times n$. Partition

$$N = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad (4.3)$$

with C of size $k \times k$. By definition, the k th generalized eigenvalues of M are the numbers $x \in \mathbb{C}$ such that

$$\det \begin{pmatrix} A & B - xI_{n-k} \\ C & D \end{pmatrix} = 0. \quad (4.4)$$

The assumption that M is totally positive implies in particular that all the entries of N are positive. We bring N to a simpler form by means of elementary row operations. Denote

$$G_j := I_n - \frac{N_{j,0}}{N_{j+1,0}} E_{j,j+1},$$

where for $j, l \in [0 : n-1]$, $N_{j,l}$ denotes the (j, l) entry of N , and where $E_{j,l}$ is the elementary matrix of size $n \times n$ whose (j, l) entry is 1 and which has all its other entries equal to zero. Multiplying N on the left with the matrix G_j amounts to subtracting from row j , $N_{j,0}/N_{j+1,0}$ times row $j+1$. This operation eliminates the $(j, 0)$ entry of N .

We also define

$$\tilde{G}_j := \begin{cases} I_n + \frac{N_{j,0}}{N_{j+1,0}} E_{j+k,j+k+1}, & \text{if } j+k+1 < n, \\ I_n, & \text{otherwise.} \end{cases}$$

The matrices G_j and \tilde{G}_j satisfy

$$G_j \begin{pmatrix} 0 & I_{n-k} \\ 0 & 0 \end{pmatrix} \tilde{G}_j = \begin{pmatrix} 0 & I_{n-k} \\ 0 & 0 \end{pmatrix}, \quad j \in [0 : n-2], \quad (4.5)$$

where we use the same decomposition in blocks as in (4.3).

Consider the transformed matrix

$$\tilde{N}^{(1)} := G_{n-2} \dots G_1 G_0 N \tilde{G}_0 \tilde{G}_1 \dots \tilde{G}_{n-2}. \quad (4.6)$$

The matrix $\tilde{N}^{(1)}$ has all its entries in the first column equal to zero except for the last one, which equals $N_{n-1,0}$. Let $N^{(1)}$ be the matrix obtained by removing the first column and the last row of $\tilde{N}^{(1)}$. Using (4.5), we deduce that the k th generalized eigenvalues of M are the points $x \in \mathbb{C}$ such that

$$\det \begin{pmatrix} \tilde{A} & \tilde{B} - xI_{n-k} \\ \tilde{C} & \tilde{D} \end{pmatrix} = 0, \quad (4.7)$$

where

$$N^{(1)} = \begin{pmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{pmatrix}$$

with \tilde{C} of size $(k-1) \times (k-1)$. Observe that compared to (4.4), the diagonal of x 's in (4.7) is closer to the main diagonal.

We claim that the matrix $-N^{(1)}$ is a TP matrix (note the minus sign). For convenience we label the rows and columns of $N^{(1)}$ from 0 to $n-2$ and from 1 to $n-1$, respectively. Let $K = \{r, r-1, \dots, r-q+1\} \subset [0 : n-2]$ and $L = \{c, c-1, \dots, c-q+1\} \subset [1 : n-1]$ be two index sets. From the fact that N is TP we have that

$$\det N(K \cup \{r+1\}, L \cup \{0\}) > 0,$$

i.e., the determinant of the enlarged submatrix obtained by adjoining row $r+1$ and column 0 to $N(K, L)$ is positive.

Define

$$\widehat{N} := G_r \dots G_1 G_0 N.$$

It is clear that

$$0 < \det N(K \cup \{r+1\}, L \cup \{0\}) = \det \widehat{N}(K \cup \{r+1\}, L \cup \{0\}), \quad (4.8)$$

where the last equality follows since the row operations G_0, \dots, G_r applied to N leave the determinant invariant.

From the definition of the row operations G_0, \dots, G_r , the submatrix in the right hand side of (4.8) is zero in its first column except for its last entry. Expanding the determinant along its first column we therefore see that

$$\det \widehat{N}(K \cup \{r+1\}, L \cup \{0\}) = (-1)^q N_{r+1,0} \det \widehat{N}(K, L),$$

which combined with (4.8) and the TP property of N yields

$$(-1)^q \det \widehat{N}(K, L) > 0. \quad (4.9)$$

The property (4.9) remains valid with \widehat{N} replaced by the matrix

$$G_{n-2} \dots G_{r+1} \widehat{N} = G_{n-2} \dots G_1 G_0 N,$$

since the row operations G_{r+1}, \dots, G_{n-2} applied to \widehat{N} leave the submatrix indexed by rows K and columns L invariant. Since K and L are arbitrary index sets, this implies by Fekete's criterion that the matrix of size $(n-1) \times (n-1)$,

$$-(G_{n-2} \dots G_1 G_0 N)([0 : n-2], [1 : n-1]),$$

is TP. This implies in turn that

$$-\widetilde{N}^{(1)}([0 : n-2], [1 : n-1]) = -N^{(1)}$$

is also TP, since (cf. (4.6)) each of the column operations $\widetilde{G}_0, \widetilde{G}_1, \dots, \widetilde{G}_{n-2}$ adds to a column a positive multiple of the previous column; it is straightforward to verify that such operations leave the total positivity of a matrix invariant.

By repeating the transformation $N^{(0)} := N \mapsto N^{(1)}$ k times, we get a series of matrices $N^{(0)}, N^{(1)}, N^{(2)}, \dots, N^{(k)}$ so that the k th generalized eigenvalues of M are the (usual) eigenvalues of $N^{(k)}$. Moreover, the matrix $(-1)^k N^{(k)}$ is TP. One of the assertions of the Gantmacher-Krein theorem implies then the validity of the first statement of the Proposition.

Finally, if we apply to the leading principal submatrix Q of M the operations described above, and denote the resulting series of matrices by $Q^{(0)}, Q^{(1)}, Q^{(2)}, \dots, Q^{(k)}$, then $Q^{(j)}$ will be the leading principal submatrix of $N^{(j)}$ for each $j \in [0 : k]$. In particular, $Q^{(k)}$ is the leading principal submatrix of $N^{(k)}$ and the Gantmacher-Krein theorem implies the interlacing property we want. \square

Since TP matrices are dense in the class of TNN matrices, we obtain

Corollary 4.2. *(Generalized eigenvalues of TNN matrices:) Fix $0 \leq k < n$ and let $M \in \mathbb{C}^{(n+k) \times (n+k)}$ be a TNN matrix. Then the k th generalized eigenvalues of M lie in $[0, \infty)$ if k is even and lie in $(-\infty, 0]$ if k is odd. Denoting the k th generalized eigenvalues of M by $(y_i)_{i=1,2,\dots}$ and those of its principal leading submatrix by $(x_i)_{i=1,2,\dots}$, both of them ordered by increasing modulus and counting multiplicities, then we have the (weak) interlacing*

$$0 \leq |y_1| \leq |x_1| \leq |y_2| \leq |x_2| \leq \dots$$

Note that the moduli can be removed if k is even and replaced by minus signs if k is odd.

Remark 4.3. In the process of approximating a TNN matrix by a sequence of TP matrices, some of the generalized eigenvalues may escape to infinity. This will always happen for the kind of banded matrices we are interested in.

4.2 The approach of Eiermann-Varga revisited

In the proofs of Theorems 2.7 and 2.12, we will use some ideas from Eiermann-Varga [12], which we now review.

Consider $Q_n(x) = (-1)^n \det(H_n - xI_n)$ with H_n the $n \times n$ two-diagonal Hessenberg matrix in (2.7). Let $P : [0 : n-1] \mapsto [0 : n-1]$ be the permutation that sorts the indices according to their residue modulo $p+1$, in the natural way, that is,

$$(P(0), P(1), \dots, P(n-1)) = (0, p+1, 2p+2, \dots; 1, p+2, 2p+3, \dots; \dots; p, 2p+1, 3p+2, \dots).$$

Also denote with P the corresponding permutation matrix such that $P\mathbf{e}_j = \mathbf{e}_{P(j)}$ for $j \in [0 : n-1]$. Thus P has in its j th column the value 1 at position $P(j)$ and zero at all other positions. We consider the permuted matrix $P^{-1}H_nP - xI_n$. It has a block bidiagonal structure:

$$P^{-1}H_nP - xI_n = \begin{pmatrix} X_0 & Y_0 & & & \\ 0 & X_1 & Y_1 & & \\ & & \ddots & \ddots & \\ & & & X_{p-1} & Y_{p-1} \\ Y_p & & & & X_p \end{pmatrix} \quad (4.10)$$

where $X_j = -xI_{n_j}$ with $n_j := \lfloor \frac{n+p-j}{p+1} \rfloor$, where Y_j is the principal truncation of size $n_j \times n_{j+1}$ of the semi-infinite bidiagonal matrix

$$Y_{j,\infty} = \begin{pmatrix} 1 & & & & \\ a_{j+1} & 1 & & & \\ & a_{p+j+2} & 1 & & \\ & & a_{2p+j+3} & 1 & \\ & & & \ddots & \ddots \end{pmatrix} \quad (4.11)$$

for $j \in [0 : p-1]$, and where Y_p is the principal truncation of size $n_p \times n_0$ of the matrix

$$Y_{p,\infty} = \begin{pmatrix} a_0 & 1 & & & \\ & a_{p+1} & 1 & & \\ & & a_{2p+2} & 1 & \\ & & & \ddots & \ddots \end{pmatrix}. \quad (4.12)$$

Lemma 4.4. (a) Let A be a matrix of size $n \times n$ as in the right hand side of (4.10), with diagonal blocks $X_j = -xI_{n_j}$, $j \in [0 : p]$. Then we have

$$\det A = (-1)^{n-n_0} x^{n-(p+1)n_0} \det(Y_0 Y_1 \dots Y_p - x^{p+1} I_{n_0}).$$

(b) Under the same hypotheses, if we replace X_0 by an arbitrary square matrix of size $n_0 \times n_0$, then we have

$$\det A = (-1)^{n-n_0} x^{n-(p+1)n_0} \det(Y_0 Y_1 \dots Y_p + x^p X_0).$$

Proof. Use Gaussian elimination with the blocks X_1, \dots, X_p as pivots to eliminate the blocks above the main diagonal. After these operations, the block we obtain in the upper left corner is the matrix $X_0 + x^{-p} Y_0 Y_1 \dots Y_p$. The exponent of x is easily determined. \square

Note that the zeros of Q_n are the points x where $\det(P^{-1}H_nP - xI_n)$ vanishes. Now we apply Lemma 4.4(a) to this determinant. Each of the matrices Y_0, Y_1, \dots, Y_p in (4.11)–(4.12) is bidiagonal with positive entries and hence TNN. Thus also the matrix product $Y_0 Y_1 \dots Y_p$ is TNN (actually it is *oscillatory* [12]). This already shows that all the eigenvalues of $Y_0 Y_1 \dots Y_p$ are in $[0, \infty)$. Taking into account the factor x^{p+1} in Lemma 4.4(a), we then see that the zeros of Q_n are all located on the star S_+ .

Carrying on this approach a little further and using the Gantmacher-Krein theory, one obtains the (strict) interlacing relations for the zeros of the polynomials Q_n and Q_{n+1} , and for Q_n and Q_{n+p+1} . See Eiermann-Varga [12]. Alternative proofs of the interlacing are in [16] and [24].

4.3 Proofs of Theorems 2.7 and 2.12

In this section we prove Theorems 2.7 and 2.12. To this end we will rely on Cor. 4.2 and the ideas in Section 4.2.

We always label rows and columns starting from 0. We will assume throughout the proof that n is a fixed multiple of $p+1$ and we fix $k \in [0 : p-1]$. The modifications if n is not a multiple of $p+1$ are discussed at the end of this section.

4.3.1 Proof of Theorem 2.7(a) (n a multiple of $p+1$)

Recall that $P_{k,n}(x)$ is the determinant of the matrix obtained by skipping rows $[0 : k-1]$ and columns $[n-k : n-1]$ of $H_n - xI_n$. Applying the permutation P above, this is equivalent to skipping certain rows and columns of the permuted matrix (4.10). More precisely, $P_{k,n}(x)$ is, up to its sign, equal to the determinant of the submatrix obtained by skipping the first row of each of the blocks $X_0, Y_0, X_1, Y_1, \dots, X_{k-1}, Y_{k-1}$ in (4.10), and skipping the last column of each of the blocks $X_p, Y_{p-1}, X_{p-1}, Y_{p-2}, \dots, X_{p-k+1}, Y_{p-k}$ (here we are using the fact that n is a multiple of $p+1$). This can be seen as follows: if we write the submatrix of $H_n - xI_n$ as $L(H_n - xI_n)R$, with L and R suitable submatrices of the identity matrix I_n (of size $(n-k) \times n$ and $n \times (n-k)$, respectively), and similarly the submatrix of $P^{-1}(H_n - xI_n)P$ as $\tilde{L}P^{-1}(H_n - xI_n)P\tilde{R}$, then $\tilde{L} = P_1LP$ and $\tilde{R} = P^{-1}RP_2$, for some permutation matrices P_1 and P_2 of size $n-k$.

Due to the above skipping of rows and columns, some of the diagonal blocks X_j in (4.10) will become rectangular instead of square. Thus we cannot apply Lemma 4.4 anymore. Our goal is therefore to make all the diagonal blocks square again. More precisely, our goal is to get a matrix as in the right hand side of (4.10) with diagonal blocks $X'_j = -xI$, for the identity matrix of certain size, $j \in [1 : p]$, and with $X'_0 = \begin{pmatrix} 0 & -xI \\ 0_{k \times k} & 0 \end{pmatrix}$. (We write X'_j, Y'_j to distinguish from the blocks X_j, Y_j in (4.10).) We will then be able to apply Lemma 4.4(b).

Recall that in the determinantal formula for $P_{k,n}(x)$ we are skipping rows $[0 : k-1]$ of $H_n - xI_n$. Then in the first k columns of this matrix there is only one non-zero entry left, being a_0, \dots, a_{k-1} respectively. Expanding the determinant along the columns $[1 : k-1]$ (we do not touch column 0) we necessarily have to pick these entries. Then the determinant equals $\pm a_1 \dots a_{k-1}$ times the determinant of the matrix obtained by skipping the rows and columns in which the entries a_1, \dots, a_{k-1} are standing. These are columns $[1 : k-1]$ and rows $[p+1 : p+k-1]$.

From the skipping of rows $[p+1 : p+k-1]$, we see that in columns $[p+2 : p+k-1]$ there is only one non-zero entry left, being $a_{p+2}, \dots, a_{p+k-1}$ respectively. So again the determinant picks up a factor $\pm a_{p+2} \dots a_{p+k-1}$, and we can proceed with the determinant of the matrix obtained by skipping the rows and columns in which the entries $a_{p+2}, \dots, a_{p+k-1}$ are standing. These are columns $[p+2 : p+k-1]$ and rows $[2p+2 : 2p+k-1]$.

From the skipping of rows $[2p+2 : 2p+k-1]$, we now have only one non-zero entry left in each of the columns $[2p+3 : 2p+k-1]$, being $a_{2p+3}, \dots, a_{2p+k-1}$ respectively. We can then make a reduction as in the previous paragraphs. Carrying on this scheme a few more steps, we are left with the following submatrix of $H_n - xI_n$: It is obtained by skipping the rows

$$[0 : k-1] \cup [p+1 : p+k-1] \cup [2p+2 : 2p+k-1] \cup \dots \cup \{(k-1)p+k-1\} \quad (4.13)$$

and the columns

$$[1 : k-1] \cup [p+2 : p+k-1] \cup [2p+3 : 2p+k-1] \cup \dots \cup \{(k-2)p+k-1\} \quad (4.14)$$

in the starting matrix $H_n - xI_n$.

We can do similar operations with the last rows and columns of $H_n - xI_n$. Indeed, recall that in the definition of $P_{k,n}(x)$ we are skipping columns $[n-k : n-1]$ of $H_n - xI_n$. The determinant can then be further reduced to the one obtained by skipping the rows

$$\{n - (k-1)p - k\} \cup \dots \cup [n-2p-k : n-2p-3] \cup [n-p-k : n-p-2] \cup [n-k : n-1] \quad (4.15)$$

and the columns

$$\{n - kp - k\} \cup \dots \cup [n-2p-k : n-2p-2] \cup [n-p-k : n-p-1] \cup [n-k : n-1] \quad (4.16)$$

in the matrix $H_n - xI_n$.

Summarizing, we see that $P_{k,n}(x)$ is, up to a constant, equal to the determinant of the submatrix of $H_n - xI_n$ obtained by skipping the rows (4.13) and (4.15) and the columns (4.14) and (4.16).

The skipping of the indicated rows and columns of $H_n - xI_n$ is again equivalent (up to a sign) to removing certain rows and columns of the permuted matrix $P^{-1}H_nP - xI_n$ in (4.10). This leads to the formula

$$P_{k,n}(x) = c \det \begin{pmatrix} X'_0 & Y'_0 & & & \\ 0 & X'_1 & Y'_1 & & \\ & & \ddots & \ddots & \\ & & & X'_{p-1} & Y'_{p-1} \\ Y'_p & & & & X'_p \end{pmatrix}, \quad (4.17)$$

$c \neq 0$, where X'_0 is obtained by skipping the first k rows and last k columns of X_0 ; where X'_j , $j \in [1 : p]$, is obtained by skipping the first $\max\{k-j, 0\}$ rows and columns and the last $\max\{j-p+k, 0\}$ rows and columns of X_j ; where Y'_j , $j \in [0 : p-1]$, is obtained by skipping the first $\max\{k-j, 0\}$ rows and $\max\{k-j-1, 0\}$ columns and the last $\max\{j-p+k, 0\}$ rows and $\max\{j-p+k+1, 0\}$ columns of Y_j ; and finally Y'_p is obtained by skipping the last k rows and columns of Y_p .

Note that each of the diagonal blocks X'_j , $j \in [1 : p]$ in (4.17) is of the form $-xI$ and moreover

$$X'_0 = \begin{pmatrix} 0 & -xI \\ 0_{k \times k} & 0 \end{pmatrix} \quad (4.18)$$

with k zero columns added at the left and k zero rows at the bottom. Hence we are in a position to apply Lemma 4.4(b): this yields

$$P_{k,n}(x) = cx^{k(p-k)} \det \left(Y'_0 Y'_1 \dots Y'_p - x^{p+1} \begin{pmatrix} 0 & I \\ 0_{k \times k} & 0 \end{pmatrix} \right), \quad (4.19)$$

$c \neq 0$. Note that each of the matrices Y'_0, Y'_1, \dots, Y'_p in (4.17) is bidiagonal with nonnegative entries and hence TNN. Thus also the matrix product $Y'_0 Y'_1 \dots Y'_p$ is TNN. Cor. 4.2 and (4.19) then imply that all the zeros of $P_{k,n}$ lie on the star S_+ (S_-) if k is even (odd). Finally, if we apply the Cauchy-Binet formula to $\det(Y'_0 Y'_1 \dots Y'_p)$ then we see that this determinant is the sum of a finite number of nonnegative terms with at least one term strictly positive (for instance, the term obtained by multiplying the determinants of the principal leading submatrices of Y'_i , $i \in [0 : p]$, is strictly positive). Noting that $m_{k,n} = k(p-k)$ if n is a multiple of $p+1$, we now obtain Theorem 2.7(a).

4.3.2 Proof of Theorem 2.7(b) (n a multiple of $p+1$)

Now we will prove the interlacing between the zeros of $P_{k,n}$ and $P_{k,n+1}$ in Theorem 2.7(b), still assuming that n is a multiple of $p+1$.

Recall that in the determinantal formula for $P_{k,n+1}(x)$ we are skipping the rows $[0 : k-1]$ of $H_{n+1} - xI_{n+1}$. In exactly the same way as in Section 4.3.1, this leads to an iterated skipping process, allowing us to skip the rows (4.13) and the columns (4.14) of $H_{n+1} - xI_{n+1}$.

In the definition of $P_{k,n+1}(x)$ we are skipping the columns $[n-k+1 : n]$ of $H_{n+1} - xI_{n+1}$. This leads again to an iterated skipping process, allowing us to skip the rows

$$\{n-(k-2)p-k+1\} \cup \dots \cup [n-2p-k+1 : n-2p-3] \cup [n-p-k+1 : n-p-2] \cup [n-k+1 : n-1] \quad (4.20)$$

and the columns

$$\{n-(k-1)p-k+1\} \cup \dots \cup [n-2p-k+1 : n-2p-2] \cup [n-p-k+1 : n-p-1] \cup [n-k+1 : n] \quad (4.21)$$

of $H_{n+1} - xI_{n+1}$. Note that we are *not* skipping the rows $n, n-p-1, n-2p-2, \dots$ and the columns $n-p, n-2p-1, n-3p-2, \dots$, although we are allowed to do that. The reason for not skipping these rows and columns, is because that would complicate the comparison to the formulas (4.15)–(4.16) for $P_{k,n}$.

In terms of the permuted matrix (4.17) we obtain

$$P_{k,n+1}(x) = c \det \begin{pmatrix} \tilde{X}_0 & \tilde{Y}_0 & & & \\ 0 & \tilde{X}_1 & \tilde{Y}_1 & & \\ & & \ddots & \ddots & \\ & & & \tilde{X}_{p-1} & \tilde{Y}_{p-1} \\ \tilde{Y}_p & & & & \tilde{X}_p \end{pmatrix}, \quad (4.22)$$

$c \neq 0$, where the blocks \tilde{X}_j, \tilde{Y}_j are obtained from the blocks X'_j, Y'_j for $P_{k,n}$ in (4.17) by the formulas

$$\begin{aligned} \tilde{X}_0 &= \begin{pmatrix} X'_0 & -x\tilde{\mathbf{e}} \\ 0 & 0 \end{pmatrix}, \\ \tilde{X}_j &= X'_j, & j \in [1 : p-k], \\ \tilde{X}_j &= \begin{pmatrix} X'_j & 0 \\ 0 & -x \end{pmatrix}, & j \in [p-k+1 : p], \end{aligned}$$

with $\tilde{\mathbf{e}}$ denoting the k th last column of the identity matrix, and

$$\begin{aligned}\tilde{Y}_0 &= \begin{pmatrix} Y'_0 \\ a_* \mathbf{e}^T \end{pmatrix}, & \tilde{Y}_j &= Y'_j, & j &\in [1 : p-k-1], \\ \tilde{Y}_{p-k} &= (Y'_{p-k} \quad \mathbf{e}), & \tilde{Y}_j &= \begin{pmatrix} Y'_j & \mathbf{e} \\ 0 & a_* \end{pmatrix}, & j &\in [p-k+1 : p],\end{aligned}$$

with \mathbf{e} denoting the last column of the identity matrix, and with the a_* certain recurrence coefficients. Lemma 4.4(b) yields

$$P_{k,n+1}(x) = cx^{(k-1)(p-k)} \det \left(\tilde{Y}_0 \tilde{Y}_1 \dots \tilde{Y}_p - x^{p+1} \begin{pmatrix} 0 & I \\ 0_{k \times k} & 0 \end{pmatrix} \right), \quad (4.23)$$

$c \neq 0$. But the cyclic product $\tilde{Y}_0 \tilde{Y}_1 \dots \tilde{Y}_p$ has $Y'_0 Y'_1 \dots Y'_p$ as its leading principal submatrix. Hence Cor. 4.2 yields the required interlacing relation in Theorem 2.7(b). It is also easy to see, as at the end of Section 4.3.1, that $\det(\tilde{Y}_0 \tilde{Y}_1 \dots \tilde{Y}_p) > 0$.

4.3.3 Proof of Theorem 2.7(c) (n a multiple of $p+1$)

Next we prove the interlacing between the zeros of $P_{k,n}$ and $P_{k,n+p+1}$ in Theorem 2.7(c), still assuming that n is a multiple of $p+1$. Observe that $n+p+1$ is also a multiple of $p+1$. Applying exactly the same approach as in Section 4.3.1, we find that $P_{k,n+p+1}$ can be written as in the right hand side of (4.22), where the blocks \tilde{X}_j, \tilde{Y}_j are now obtained from the blocks X'_j, Y'_j for $P_{k,n}$ in (4.17) by the formulas

$$\tilde{X}_0 = \begin{pmatrix} X'_0 & -x\tilde{\mathbf{e}} \\ 0 & 0 \end{pmatrix}, \quad \tilde{X}_j = \begin{pmatrix} X'_j & 0 \\ 0 & -x \end{pmatrix}, \quad j \in [1 : p],$$

with again $\tilde{\mathbf{e}}$ denoting the k th last column of the identity matrix, and

$$\begin{aligned}\tilde{Y}_j &= \begin{pmatrix} Y'_j & 0 \\ a_* \mathbf{e}^T & 1 \end{pmatrix}, & j &\in [0 : p-k-1], \\ \tilde{Y}_j &= \begin{pmatrix} Y'_j & \mathbf{e} \\ 0 & a_* \end{pmatrix}, & j &\in [p-k : p],\end{aligned}$$

with \mathbf{e} denoting the last column of the identity matrix and with a_* certain recurrence coefficients. We can again apply Lemma 4.4(b). But the cyclic product $\tilde{Y}_0 \tilde{Y}_1 \dots \tilde{Y}_p$ has $Y'_0 Y'_1 \dots Y'_p$ as its leading principal submatrix. Cor. 4.2 then yields the required interlacing relation in Theorem 2.7(c).

4.3.4 Proof of Theorem 2.12 (n a multiple of $p+1$)

Next we prove the interlacing relations between the zeros of $P_{k,n}$ and $P_{k,l,n}$ in Theorem 2.12, assuming that $1 \leq k < l \leq p$ and n is a multiple of $p+1$. Recall that $P_{k,l,n}$ is the determinant of the submatrix obtained by skipping the rows $[0 : k-1]$ and the columns $\{n-l\} \cup [n-k+1 : n-1]$ of $H_n - xI_n$. This leads to an iterated skipping process, allowing us to skip the rows (4.13) and

$$\{n-(k-2)p-(k-1)\} \cup \dots \cup [n-2p-k+1 : n-2p-3] \cup [n-p-k+1 : n-p-2] \cup [n-k+1 : n-1],$$

and the columns (4.14) and

$$\{n-(k-1)p-k+1\} \cup \dots \cup [n-2p-k+1 : n-2p-2] \cup [n-p-k+1 : n-p-1] \cup \{n-l\} \cup [n-k+1 : n-1]$$

in the matrix $H_n - xI_n$. Then $P_{k,l,n}$ can be written as in the right hand side of (4.22), where the blocks \tilde{X}_j, \tilde{Y}_j are now obtained from the blocks X'_j, Y'_j for $P_{k,n}$ in (4.17) by the formulas

$$\begin{aligned}\tilde{X}_0 &= \begin{pmatrix} X'_0 & -x\tilde{\mathbf{e}} \end{pmatrix}, & \tilde{X}_j &= X'_j, & j &\in [1 : p-k], j \neq p-l+1, \\ X'_{p-l+1} &= \begin{pmatrix} \tilde{X}_{p-l+1} & -x\mathbf{e} \end{pmatrix}, & \tilde{X}_j &= \begin{pmatrix} X'_j & 0 \\ 0 & -x \end{pmatrix}, & j &\in [p-k+1 : p],\end{aligned}$$

and

$$\begin{aligned}\tilde{Y}_j &= Y'_j, & j &\in [0 : p-k-1], j \neq p-l, \\ Y'_{p-l} &= \begin{pmatrix} \tilde{Y}_{p-l} & \mathbf{e} \end{pmatrix}, \\ \tilde{Y}_{p-k} &= \begin{pmatrix} Y'_{p-k} & \mathbf{e} \end{pmatrix}, \\ \tilde{Y}_j &= \begin{pmatrix} Y'_j & \mathbf{e} \\ 0 & a_* \end{pmatrix}, & j &\in [p-k+1 : p],\end{aligned}$$

with the notations $\mathbf{e}, \tilde{\mathbf{e}}, a_*$ as defined before.

There is now a complication for the block \tilde{X}_{p-l+1} : this is a scalar multiple of the identity matrix but with the last column skipped. Moreover, \tilde{Y}_{p-l} is a submatrix of Y'_{p-l} rather than the other way around. We will resolve both issues by appending an extra row and column at the end of some of the blocks in the matrix in the right hand side of (4.22). These extra rows and columns will have a triangular structure and therefore will not influence the determinant (except for a scalar factor), as we will see below. Here is the definition: we put

$$\hat{X}_{p-l+1} := \begin{pmatrix} \tilde{X}_{p-l+1} & -x\mathbf{e} \end{pmatrix},$$

thereby making this block square again. We also put

$$\hat{X}_0 := \begin{pmatrix} \tilde{X}_0 \\ 0 \end{pmatrix}, \quad \hat{X}_j := \begin{pmatrix} \tilde{X}_j & 0 \\ 0 & -x \end{pmatrix}, \quad j \in [1 : p-l],$$

with the zeros denoting a row or column vector,

$$\hat{Y}_j := \begin{pmatrix} \tilde{Y}_j & 0 \\ 0 & 1 \end{pmatrix}, \quad j \in [0 : p-l-1], \quad \hat{Y}_{p-l} := \begin{pmatrix} \tilde{Y}_{p-l} & \mathbf{e} \\ 0 & a \end{pmatrix},$$

for an arbitrary constant $a \neq 0$, and

$$\begin{aligned}\hat{X}_j &:= \tilde{X}_j, & j &\in [p-l+2 : p], \\ \hat{Y}_j &:= \tilde{Y}_j, & j &\in [p-l+1 : p].\end{aligned}$$

As mentioned, the triangular structure of the added rows and columns implies that

$$\det \begin{pmatrix} \hat{X}_0 & \hat{Y}_0 & & & \\ 0 & \hat{X}_1 & \hat{Y}_1 & & \\ & & \ddots & \ddots & \\ & & & \hat{X}_{p-1} & \hat{Y}_{p-1} \\ \hat{Y}_p & & & & \hat{X}_p \end{pmatrix} = \pm a \det \begin{pmatrix} \tilde{X}_0 & \tilde{Y}_0 & & & \\ 0 & \tilde{X}_1 & \tilde{Y}_1 & & \\ & & \ddots & \ddots & \\ & & & \tilde{X}_{p-1} & \tilde{Y}_{p-1} \\ \tilde{Y}_p & & & & \tilde{X}_p \end{pmatrix}. \quad (4.24)$$

(To see this, expand the determinant on the left-hand side of (4.24) along the last row of \widehat{X}_0 and \widehat{Y}_0 . This row has only one nonzero entry. This allows to delete this row and also the last column of \widehat{Y}_0 and \widehat{X}_1 . Next we expand the new determinant along the last row of \widehat{X}_1 and \widehat{Y}_1 , and so on.) So we can work with $\widehat{X}_j, \widehat{Y}_j$ instead of $\widetilde{X}_j, \widetilde{Y}_j$.

Summarizing, $P_{k,l,n}$ can be written as a constant times the left hand side of (4.24). Combining the above descriptions, we see that the blocks $\widehat{X}_j, \widehat{Y}_j$ are obtained from the blocks X'_j, Y'_j for $P_{k,n}$ in (4.17) by the formulas

$$\widehat{X}_0 = \begin{pmatrix} X'_0 & -x\mathbf{e} \\ 0 & 0 \end{pmatrix}, \quad \widehat{X}_j = \begin{pmatrix} X'_j & 0 \\ 0 & -x \end{pmatrix}, \quad j \in [1 : p-l] \cup [p-k+1 : p],$$

$$\widehat{X}_j = X'_j, \quad j \in [p-l+1 : p-k],$$

and

$$\widehat{Y}_j = \begin{pmatrix} Y'_j & 0 \\ 0 & 1 \end{pmatrix}, \quad j \in [0 : p-l-1],$$

$$\widehat{Y}_{p-l} = \begin{pmatrix} Y'_{p-l} \\ a\mathbf{e}^T \end{pmatrix},$$

$$\widehat{Y}_j = Y'_j, \quad j \in [p-l+1 : p-k-1],$$

$$\widehat{Y}_{p-k} = (Y'_{p-k} \quad \mathbf{e}),$$

$$\widehat{Y}_j = \begin{pmatrix} Y'_j & \mathbf{e} \\ 0 & a_* \end{pmatrix}, \quad j \in [p-k+1 : p].$$

Lemma 4.4(b) can be applied and yields

$$P_{k,l,n}(x) = cx^{k(p-k)+k-l} \det \left(\widehat{Y}_0 \widehat{Y}_1 \dots \widehat{Y}_p - x^{p+1} \begin{pmatrix} 0 & I \\ 0_{k \times k} & 0 \end{pmatrix} \right), \quad (4.25)$$

$c \neq 0$. But the principal leading submatrix of $\widehat{Y}_0 \widehat{Y}_1 \dots \widehat{Y}_p$ is precisely the matrix $Y'_0 Y'_1 \dots Y'_p$. Taking into account that $m_{k,n} = k(p-k)$ if n is a multiple of $p+1$, we then obtain Theorem 2.12.

4.3.5 Modifications if n is not a multiple of $p+1$

If n is not a multiple of $p+1$, we can use the same ideas but with a few modifications. We focus on the construction for $P_{k,n}$ in Section 4.3.1. We can again write $P_{k,n}$ as in (4.17). But now the description of the blocks X'_j, Y'_j depends on the residue q of n modulo $p+1$, $q \in [1 : p]$. More precisely, the number of rows and columns to be skipped at the top and at the left of each block X'_j, Y'_j , is exactly the same as in Section 4.3.1. But for the rows and columns at the bottom and at the right of each block, the description that was given for X'_j, Y'_j in Section 4.3.1 should now be applied to X'_{j+q}, Y'_{j+q} (where we view the subscripts modulo $p+1$).

The above description implies in particular that $X'_0 = \begin{pmatrix} 0 & -xI \end{pmatrix}$, with k zero columns added at its left, and $X'_q = \begin{pmatrix} -xI \\ 0 \end{pmatrix}$, with k zero rows added at its bottom. All the other X_j are of the form $-xI$. Thus we are not able to apply Lemma 4.4.

To get around this issue, we append k extra rows and columns in the top and/or left part of some of the blocks. We set

$$\widetilde{X}_0 = \begin{pmatrix} -xE_k \\ X'_0 \end{pmatrix}, \quad \widetilde{X}_j = \begin{pmatrix} -xI_k & 0 \\ 0 & X'_j \end{pmatrix}, \quad j \in [1 : q-1],$$

$$\widetilde{X}_q = \begin{pmatrix} 0 & X'_q \end{pmatrix}, \quad \widetilde{X}_j = X'_j, \quad j \in [q+1 : p],$$

where E_k is the submatrix formed by the first k rows of the identity matrix, and

$$\begin{aligned}\tilde{Y}_j &= \begin{pmatrix} I_k & 0 \\ 0 & Y'_j \end{pmatrix}, & j \in [0 : q-1] \\ \tilde{Y}_j &= Y'_j, & j \in [q : p].\end{aligned}$$

Due to the triangular structure of the added rows and columns, they leave the determinant invariant up to its sign. So we can replace the X'_j, Y'_j by the \tilde{X}_j, \tilde{Y}_j .

The blocks \tilde{X}_j are now all of the form $-xI$, except for \tilde{X}_q which is of the form $\tilde{X}_q = \begin{pmatrix} 0 & -xI \\ 0_{k \times k} & 0 \end{pmatrix}$. To bring the matrix to the form required by Lemma 4.4, it suffices to apply a cyclic block permutation. More precisely, we move each of the blocks \tilde{X}_j, \tilde{Y}_j , q positions to the top and to the left (in a cyclic way, thus reappearing at the bottom or right of the matrix when crossing the top or left matrix border respectively). We relabel the blocks in the permuted matrix as X'_j, Y'_j in the usual way. Thus $X'_j := \tilde{X}_{j+q}$ and $Y'_j := \tilde{Y}_{j+q}$ where we view the subscripts modulo $p+1$.

Summarizing, we can write $P_{k,n}$ as in (4.17), where the blocks X'_j, Y'_j have the form described in Lemma 4.4. The blocks X'_j, Y'_j will play exactly the same role as in Sections 4.3.1–4.3.4.

The above construction for $P_{k,n}$, can also be used for the matrices associated to $P_{k,n+1}$, $P_{k,n+p+1}$ and $P_{k,l,n}$. The blocks in these matrices differ from those for $P_{k,n}$ only in their bottom right matrix corner. So the above modifications, which only affect a fixed number of *top* and *left* rows and columns in each block, are exactly the same for each of these matrices. Since the interlacing relations are obtained by comparing the *bottom* and *right* rows and columns, the proofs in Sections 4.3.2–4.3.4 then carry on in exactly the same way as before.

The only effect of adding rows and columns at the top or at the left, is that it changes the sizes of some of the blocks X'_j, Y'_j . The sizes depend explicitly on the residue q of n modulo $p+1$. The same then holds for the exponents of x yielded by Lemma 4.4. The details are straightforward.

Remark 4.5. To obtain the values for $m_{k,n}$ in (2.16) and the fact that $\tilde{P}_{k,n}(0) \neq 0$ in Theorem 2.7 with the above approach, one has to do a careful bookkeeping. Another approach for checking these statements is to use (5.6) below. It suffices there to find the patterns s that yield the lowest exponent of x in (5.6), which is a combinatorial exercise.

4.4 Interlacing for arbitrary Riemann-Hilbert minors

The techniques in Sections 4.3.4 and 4.3.5 can be used to prove the following result for arbitrary Riemann-Hilbert minors. It generalizes Theorem 2.12.

Theorem 4.6. *Consider the two-diagonal Hessenberg matrix H_n in (2.7), where n is sufficiently large. Let $k \in [0 : p-1]$ and $\kappa \in [0 : k-1]$ and consider the polynomials $P^{\mathbf{n}_1}(x)$ and $P^{\mathbf{n}_2}(x)$ (cf. (3.3)), with*

$$\begin{aligned}\mathbf{n}_1 &= (n_0, \dots, n_\kappa, n-k+\kappa+1, \dots, n-1, n), \\ \mathbf{n}_2 &= (n_0, \dots, n_{\kappa-1}, n-k+\kappa, \dots, n-1, n),\end{aligned}\tag{4.26}$$

where

$$n-p \leq n_0 < n_1 < \dots < n_\kappa < n-k+\kappa,\tag{4.27}$$

and the last $k-\kappa$ ($k-\kappa+1$) components of \mathbf{n}_1 (\mathbf{n}_2) are taken consecutively.

(a) We have

$$P^{\mathbf{n}_1}(x) = x^m \tilde{P}^{\mathbf{n}_1}(x^{p+1}), \quad P^{\mathbf{n}_2}(x) = x^{m+n_\kappa-(n-k+\kappa)} \tilde{P}^{\mathbf{n}_2}(x^{p+1}),$$

for some explicit m depending only on the residues of the indices n_0, \dots, n_κ, n modulo $p+1$. The zeros of the above polynomials \tilde{P} lie in \mathbb{R}_+ (\mathbb{R}_-) if k is even (odd).

(b) Denote by $(y_i)_{i=1,2,\dots}$ and $(x_i)_{i=1,2,\dots}$ the roots of $\tilde{P}^{\mathbf{n}_1}(x)$ and $\tilde{P}^{\mathbf{n}_2}(x)$ respectively, counting multiplicities and ordered by increasing modulus. We have the weak interlacing relation

$$0 \leq |y_1| \leq |x_1| \leq |y_2| \leq |x_2| \leq \dots$$

5 Normal family estimates

The goal of this section is to prove the following result.

Lemma 5.1. (Normal family:) Let $k \in [0 : p]$ be fixed, and assume there exists an absolute constant $R > 0$ so that

$$R^{-1} < a_n < R, \quad n \geq 0. \quad (5.1)$$

Then for any compact set $\mathcal{K} \subset \mathbb{C} \setminus S_+$ (if k is even) or $\mathcal{K} \subset \mathbb{C} \setminus S_-$ (if k is odd) and for any fixed set of indices $i_j \in \mathbb{Z}$, $j \in [0 : k]$, with

$$i_0 < i_1 < \dots < i_k \leq i_0 + p, \quad (5.2)$$

there exists a constant $M > 0$ so that for all $x \in \mathcal{K}$ and for all n we have

$$M^{-1} < \left| P^{(n+i_0, n+i_1, \dots, n+i_k)}(x) / P_{k,n}(x) \right| < M. \quad (5.3)$$

Lemma 5.1 is proved in Section 5.2. In the proof we will need a combinatorial expansion of the generalized eigenvalue determinant $P^{(n_0, n_1, \dots, n_k)}$, to which we turn now.

5.1 Combinatorial expansion of generalized eigenvalue determinants

We start with a definition.

Definition 5.2. Let $p \in \mathbb{N}$, $k \in [0 : p]$ and $n - p \leq n_0 < n_1 < \dots < n_k = n$ be fixed numbers. A *pattern* is a sequence $s = (s_j)_{j=0}^{n-1}$ such that $s_j \in \{0, 1\}$ for all j , with boundary conditions

$$s_0 = \dots = s_{k-1} = 1, \quad (5.4)$$

$$s_j = \begin{cases} 1, & j = n_0, n_1, \dots, n_{k-1}, \\ 0, & j \in [n-p : n-1] \setminus \{n_0, n_1, \dots, n_{k-1}\}, \end{cases} \quad (5.5)$$

and such that the following rule holds:

Pattern rule: For each $j \in [0 : n-p-1]$ with $s_j = 1$, exactly k out of the p numbers s_{j+1}, \dots, s_{j+p} are equal to 1.

We denote with \mathcal{S} the set of all such patterns s .

For example, if $p = 4$, $k = 2$ and $(n_0, n_1, n_2 = n) = (14, 15, 16)$ then the following sequence is a pattern: $(s_j)_{j=0}^{15} = (1, 1, 0, 1, 0, 1, 1, 0, 0, 1, 1, 1, 0, 0, 1, 1)$. For instance, note that $s_3 = 1$ and exactly 2 out of the 4 numbers $\{s_4, s_5, s_6, s_7\}$ are equal to 1, namely the numbers s_5 and s_6 .

For a pattern $s \in \mathcal{S}$, write $|s| = \sum_{j=0}^{n-p-1} s_j$. Thus $|s|$ is the number of indices $j \in [0 : n-p-1]$ for which $s_j = 1$. In the example above we have $|s| = 8$.

Remark 5.3. Let s be a pattern and consider a group of p consecutive numbers $(s_j)_{j=m-p}^{m-1}$, $p \leq m \leq n$. Def. 5.2 easily implies that $\#\{j \in [m-p : m-1] \mid s_j = 1\} \in \{k, k+1\}$.

Remark 5.4. In the case $k = 0$, we understand that there is no initial condition (5.4), and (5.5) reduces to ask $s_j = 0$ for all $j \in [n-p : n-1]$.

Proposition 5.5. *The generalized eigenvalue determinant $P^{(n_0, n_1, \dots, n_k)}$ can be written as a sum over patterns:*

$$P^{(n_0, n_1, \dots, n_k)}(x) = \sum_{s \in \mathcal{S}} (-1)^{(p-k)|s|} \left(\prod_{j=0}^{n-p-1} a_j^{s_j} \right) (-x)^{(k+1)(n-k)-(p+1)|s|-q}, \quad (5.6)$$

where $q := \sum_{j=0}^{k-1} (n+j-k-n_j) \geq 0$.

Proof. In the proof we assume that $k \geq 1$. Simpler arguments can be applied to prove (5.6) in the case $k = 0$. By definition, $P^{(n_0, n_1, \dots, n_k)}(x)$ is the determinant of the matrix obtained by skipping rows $0, \dots, k-1$ and columns n_0, \dots, n_{k-1} of the matrix $H_n - xI$, with $n = n_k$. We write this determinant as a signed sum over all permutations σ of length $n-k$:

$$P^{(n_0, n_1, \dots, n_k)}(x) = \sum_{\sigma \in S_{n-k}} (-1)^{\text{sign}(\sigma)} \left(\prod_{i=k}^{n-1} (H_n - xI)_{i, \sigma(i)} \right) \quad (5.7)$$

where $(H_n - xI)_{i,j}$ denotes the (i, j) entry of $H_n - xI$ and we view σ as a map $\sigma : [k : n-1] \rightarrow [0 : n-1] \setminus \{n_0, n_1, \dots, n_{k-1}\}$, in the natural way. We will often find it convenient to use the inverse map σ^{-1} .

Since $H_n - xI$ has only three nonvanishing diagonals, most of the terms in the sum (5.7) will be zero. To have a nonzero term we must have $\sigma^{-1}(i) \in \{i-1, i, i+p\}$ for all i . For such a σ , we define a sequence $(s_j)_{j=0}^{n-p-1}$ by $s_j = 1$ if $\sigma^{-1}(j) = j+p$ (meaning that the permutation σ selects the entry $(H_n - xI)_{j+p, j} = a_j$) and $s_j = 0$ otherwise. We define the boundary values $(s_j)_{j=n-p}^{n-1}$ as in (5.5). We claim that this sets up a bijection between the permutations σ leading to a nonzero term in (5.7), and the patterns $s \in \mathcal{S}$.

To prove this assertion, consider the matrix obtained by skipping rows $0, \dots, k-1$ of $H_n - xI$. Its leading principal submatrix of size $2p-k+1$ by $p+1$ can be partitioned in blocks as follows:

$$\begin{array}{cc} & \begin{array}{cc} k & p-k+1 \end{array} \\ \begin{array}{c} p-k \\ k \\ p-k+1 \end{array} & \begin{pmatrix} 0 & X \\ A & Y \\ 0 & B \end{pmatrix} \end{array}, \quad (5.8)$$

where $A = \text{diag}(a_0, \dots, a_{k-1})$, $B = \text{diag}(a_k, \dots, a_p)$, $X = \begin{pmatrix} -x & 1 & & \\ & \ddots & \ddots & \\ & & -x & 1 \end{pmatrix}$, and Y has its

top right entry equal to $-x$ and all its other entries equal to zero.

Observe that in each of the first k columns of (5.8) there is only one nonzero entry, being of the form a_j , $j \in [0 : k-1]$ in the block A . The permutation σ has to pick these entries. Hence $s_0 = \dots = s_{k-1} = 1$, consistent with (5.4). In particular, since we have to pick a_0 , we are not allowed to choose the entry $-x$ in the top right corner of the block Y .

Now in each of the first $p-k$ rows of (5.8), σ has to pick either the entry $-x$ or the entry 1 from the corresponding row of the block X . Since X is rectangular with one more column than row, there will be one of the chosen entries in each of the columns of X except one. This means

in turn that σ has to pick exactly one of the entries a_k, \dots, a_p of the block B in (5.8). So exactly one of the numbers s_k, \dots, s_p equals 1 and the others equal zero.

Now let $0 \leq j < i < n - p$ be two integers with $s_j = s_i = 1$ and $s_{j+1} = \dots = s_{i-1} = 0$. Assume (by induction) that exactly k out of the p numbers s_{j+1}, \dots, s_{j+p} equal 1. By the above paragraphs, this holds if $j = 0$. We will prove that exactly k out of the p numbers s_{i+1}, \dots, s_{i+p} equal 1. Applying this argument iteratively, we will then obtain that $(s_j)_{j=0}^{n-1}$ satisfies the pattern rule (Def. 5.2).

By assumption we have that exactly k of the numbers s_{j+1}, \dots, s_{j+p} equal 1. By the definition of i , this implies that exactly $k - 1$ of the numbers s_{i+1}, \dots, s_{j+p} equal 1. So it will be enough to show that exactly one of the numbers $s_{j+p+1}, \dots, s_{i+p}$ equals 1.

Consider the submatrix of $H_n - xI$ obtained by extracting rows $j + p, \dots, i + p$:

$$\left(\begin{array}{ccccccc} a_j & & & & -x & 1 & \\ & a_{j+1} & & & & -x & 1 \\ & & \ddots & & & & \\ & & & a_{i-1} & & & -x & 1 \\ & & & & a_i & & -x & 1 \end{array} \right). \quad (5.9)$$

Note that the entry $-x$ in the topmost row lies either in the same column or in a column to the right of a_i . This follows from our assumptions that exactly $k \geq 1$ of the numbers s_{j+1}, \dots, s_{j+p} equal 1, and $s_{j+1} = \dots = s_{i-1} = 0$. Now since σ picks the entries a_j and a_i , the entries $-x$ and 1 in the first and last row of (5.9) cannot be chosen. On the other hand, σ has to choose one of the entries $-x$ or 1 from each of the rows containing a_{j+1}, \dots, a_{i-1} in (5.9). Since the block formed by the entries $-x$ and 1 lying between the two horizontal lines in (5.9) is rectangular with one more column than row, there will be one of the chosen entries in each of its columns (i.e., the columns $[j + p + 1 : i + p]$), except one. This implies in turn that σ has to pick exactly one of the entries $a_{j+p+1}, \dots, a_{i+p}$. Thus exactly one of the numbers $s_{j+p+1}, \dots, s_{i+p}$ equals 1, proving the claim of the above paragraph.

In the above argument we were tacitly assuming that $[j + p + 1 : i + p]$ is disjoint from the set of skipped columns $\{n_0, n_1, \dots, n_{k-1}\}$ in the definition of $P^{(n_0, n_1, \dots, n_k)}$. If this disjointness fails, then similar arguments as above show that $[j + p + 1 : i + p]$ must contain exactly one of the indices n_0, n_1, \dots, n_{k-1} , and again, exactly one of the numbers $s_{j+p+1}, \dots, s_{i+p}$ equals 1 (recall (5.5)).

Summarizing, we have proved that each permutation σ leading to a nonzero term in (5.7) defines a pattern $s \in \mathcal{S}$ with

$$s_j = 1 \text{ if and only if } \sigma^{-1}(j) = j + p, \quad j \in [0 : n - p - 1]. \quad (5.10)$$

Conversely, we claim that each pattern $s \in \mathcal{S}$ leads to a unique permutation σ satisfying (5.10) and associated to a nonzero term in (5.7). We call σ the *permutation induced by* $s \in \mathcal{S}$. To prove its existence and uniqueness, let again $j < i$ be two numbers with $s_j = s_i = 1$, $s_{j+1} = \dots = s_{i-1} = 0$. As observed before, exactly one of the numbers $s_{j+p+1}, \dots, s_{i+p}$ equals 1; denote this number by s_{l+p} , for suitable l . Skipping the corresponding column in (5.9), the block formed by the entries $-x$ and 1 lying between the two horizontal lines in (5.9) then takes the form $\text{diag}(C, D)$ with

$$C = \begin{pmatrix} -x & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & -x \end{pmatrix}_{(l-j-1) \times (l-j-1)}, \quad D = \begin{pmatrix} 1 & & & \\ -x & \ddots & & \\ & \ddots & \ddots & \\ & & -x & 1 \end{pmatrix}_{(i-l) \times (i-l)}.$$

Note that both matrices C and D are square and triangular with nonzero diagonal entries. Hence if σ is a permutation induced by s , then we should have $\sigma^{-1}(m) = m$, for $m \in [j+p+1 : l+p-1]$ (so that σ picks the diagonal entries of C), and $\sigma^{-1}(m) = m-1$, for $m \in [l+p+1 : i+p]$ (so that σ picks the diagonal entries of D). If we follow this rule consequently for all $j < i$ with $s_j = s_i = 1$ and $s_{j+1} = \dots = s_{i-1} = 0$, and use a similar reasoning near the top left matrix corner (5.8), and near the bottom right matrix corner, then we will end up with the unique permutation σ induced by $s \in \mathcal{S}$. This proves the existence and uniqueness of σ .

Now let $s \in \mathcal{S}$ be a pattern with induced permutation σ . We claim that $\text{sign}(\sigma) = (-1)^{(p-k)|s|}$. To see this, recall that $\text{sign}(\sigma) = (-1)^K$ where K is the number of pairs of column indices (j, i) in $[0 : n-1] \setminus \{n_0, \dots, n_{k-1}\}$ with $j < i$ and $\sigma^{-1}(j) > \sigma^{-1}(i)$. Since $\sigma^{-1}(i) \in \{i-1, i, i+p\}$ for all i , in our case K is the number of pairs of column indices (j, i) with $j \in [0 : n-p-1]$, $i \in [j : j+p]$, $\sigma^{-1}(j) = j+p$ (i.e., $s_j = 1$) and $\sigma^{-1}(i) \in \{i-1, i\}$ (i.e., $s_i = 0$). But if $j \in [0 : n-p-1]$ is such that $s_j = 1$ then exactly $p-k$ of the numbers s_{j+1}, \dots, s_{j+p} are 0, by Def. 5.2. Thus $K = (p-k)|s|$ and $\text{sign}(\sigma) = (-1)^{(p-k)|s|}$, proving our claim.

Let again $s \in \mathcal{S}$ be a pattern with induced permutation σ . Let a, b, c be the number of indices $i \in [k : n-1]$ with $\sigma(i) = i-p$, $\sigma(i) = i$ and $\sigma(i) = i+1$, respectively. Thus a, b, c denote the number of entries of $H_n - xI$ of the form a_j , $-x$ and 1, respectively, that are picked by σ . We have the two relations

$$a + b + c = n - k, \quad pa - c + \sum_{i=0}^{k-1} (i - n_i) = 0. \quad (5.11)$$

The first relation is obvious. The second one follows from $\sum_{i=k}^{n-1} (i - \sigma(i)) + \sum_{i=0}^{k-1} (i - n_i) = 0$, due to the facts that σ is a permutation and we are skipping the rows $0, 1, \dots, k-1$ and columns n_0, n_1, \dots, n_{k-1} of $H_n - xI$. Now by adding the two relations in (5.11), we obtain $b = (k+1)(n-k) - (p+1)a - q$ with q as in the statement of the proposition. This yields the exponent of $-x$ in (5.6). Putting together all the above observations, we obtain (5.6). \square

Next we state a technical lemma on the existence of patterns with prescribed initial part.

Lemma 5.6. *Let p, k, n and $K \geq (p+1)(k+1) + pk$ be positive integers and let $(s_j)_{j=0}^{n-K}$ satisfy (5.4), with $s_j \in \{0, 1\}$ for all j , and such that the pattern rule holds for all $j \in [0 : n-K-p]$. Then for any indices $(n_j)_{j=0}^k$ with $n-p \leq n_0 < n_1 < \dots < n_k = n$ one can assign the numbers $(s_j)_{j=n-K+1}^{n-1}$ such that $(s_j)_{j=0}^{n-1}$ is a pattern with respect to these indices (Def. 5.2).*

Proof. We will assume that $K = \tilde{K} := (p+1)(k+1) + pk$; the case where $K > \tilde{K}$ is discussed at the end of the proof.

Consider the group of p consecutive numbers $(s_j)_{m-p+1}^m$ with $m = n-K$. Remark 5.3 implies that it has exactly k or $k+1$ entries equal to 1. Assume these entries are at the positions $m-p+i_j$, $j \in [l : k]$, with $l \in \{0, 1\}$ and $1 \leq i_l < \dots < i_k \leq p$. We define the next group of p consecutive numbers $(s_j)_{m+1}^{m+p}$ such that it has precisely k entries equal to 1, standing at the positions $m+i_j$, $j \in [1 : k]$. This definition is valid since it satisfies the pattern rule.

Next, we define $(s_j)_{m+p+1}^{m+2p}$ such that it has 1's precisely at the positions $m+p+\tilde{i}_j$ with $\tilde{i}_1 = 1$ and $\tilde{i}_j = i_j$ for $j \in [2 : k]$. In the next group $(s_j)_{m+2p+1}^{m+3p}$ we put 1's at the indices $m+2p+\hat{i}_j$ with $\hat{i}_1 = 1$, $\hat{i}_2 = 2$ and $\hat{i}_j = i_j$, $j \in [3 : k]$. We repeat this procedure until we arrive at $(s_j)_{m+(k+1)p}^{m+(k+1)p+p}$ with 1's at its first k positions and zeros elsewhere. We also define each of the numbers $(s_j)_{m+(k+1)p+1}^{m+(k+1)p+k}$ as 1. These definitions are compatible with the pattern rule.

Next, we use a similar strategy to arrive at the prescribed boundary conditions (5.5). Setting $\tilde{m} := m + (k+1)(p+1)$, we know from the last paragraph that the group $(s_j)_{\tilde{m}-p}^{\tilde{m}-1}$ has 1's at its last k positions and zeros elsewhere. Then we define $(s_j)_{\tilde{m}}^{\tilde{m}+p-1}$ with 1's at the position $\tilde{m} + p + n_0 - n$ and at its last $k-1$ positions. Next, we define $(s_j)_{\tilde{m}+p}^{\tilde{m}+2p-1}$ with 1's at the positions $\tilde{m} + 2p + n_0 - n$, $\tilde{m} + 2p + n_1 - n$ and at its last $k-2$ positions. Repeating this process, we end up with $(s_j)_{\tilde{m}+(k-1)p}^{\tilde{m}+kp-1}$ having 1's at the positions $\tilde{m} + kp + n_j - n$, $j \in [0 : k-1]$, and zeros elsewhere. These definitions are compatible with the pattern rule. Moreover, one checks that

$$\tilde{m} + kp + n_j - n = n_j, \quad \tilde{m} + kp - 1 = n - 1.$$

So we obtain the desired boundary condition (5.5).

Finally, if $K > \tilde{K} := (p+1)(k+1) + pk$ then we arbitrarily assign the numbers $(s_j)_{j=n-K+1}^{n-\tilde{K}}$ so that the pattern rule is satisfied. We then use the extended sequence $(s_j)_{j=0}^{n-\tilde{K}}$ and proceed in exactly the same way as before. \square

5.2 Proof of Lemma 5.1

We write (5.6) in the form

$$P^{(n_0, n_1, \dots, n_k)}(x) = (-x)^{(k+1)(n-k)-q} \sum_{s \in \mathcal{S}} (-1)^{(k+1)|s|} \left(\prod_{j=0}^{n-p-1} a_j^{s_j} \right) y^{-|s|}, \quad (5.12)$$

with $y := x^{p+1}$. Suppose that k is odd and $y = x^{p+1} \in \mathbb{R}_+$. Then in the above sum, each term is real and positive and hence no cancelation can occur. The same happens if k is even and $y = x^{p+1} \in \mathbb{R}_-$.

Consider the ratio of polynomials in (5.3). Both the numerator and denominator can be written as in (5.12). Let $s = (s_j)_{j=0}^{n+i_k-1}$ be a pattern corresponding to the indices $n+i_0, \dots, n+i_k$. Lemma 5.6 implies that there exists a pattern $\tilde{s} = (\tilde{s}_j)_{j=0}^{n-1}$ corresponding to the indices $[n-k+1 : n]$ such that $\tilde{s}_j = s_j$ for all $j \in [0 : n-K]$, with $K = \max\{(k+1)(p+1) + kp, -i_k + 1\}$. Clearly there are only finitely many such patterns \tilde{s} (or s) with prescribed initial part $(\tilde{s}_j)_{j=0}^{n-K}$ (or $(s_j)_{j=0}^{n-K}$). Now for each fixed $y \in \mathbb{C} \setminus \{0\}$ there exists a constant $M > 0$ so that

$$M^{-1} < \left| \left(y^{-|s|} \prod_{j=0}^{n+i_k-p-1} a_j^{s_j} \right) / \left(y^{-|\tilde{s}|} \prod_{j=0}^{n-p-1} a_j^{\tilde{s}_j} \right) \right| < M, \quad (5.13)$$

uniformly in n . This is because the products in the numerator and denominator are equal except for at most a finite number (independent of n) of factors a_j and y , and in view of (5.1). We conclude that for each fixed $x \in S_+ \setminus \{0\}$ (if k is odd) or $x \in S_- \setminus \{0\}$ (if k is even) there is a (new) constant $M > 0$ so that

$$M^{-1} < \left| P^{(n+i_0, n+i_1, \dots, n+i_k)}(x) / P_{k,n}(x) \right| < M, \quad (5.14)$$

uniformly in n . This is due to the termwise estimate (5.13) and our earlier observation that the terms in (5.12) are all real with fixed sign. This already gives us a normal family estimate on compact sets of $S_+ \setminus \{0\}$ (if k is odd) or $S_- \setminus \{0\}$ (if k is even).

To obtain the full statement of Lemma 5.1, we recall the interlacing relation for the generalized eigenvalues in Theorem 4.6. With the notations $\mathbf{n}_1, \mathbf{n}_2$ as in (4.26), Theorem 4.6 yields the partial

fraction decomposition

$$\tilde{P}^{n_2}(x)/\tilde{P}^{n_1}(x) = \alpha_0 + \sum_{i=1,2,3,\dots} \alpha_i/(x - y_i), \quad (5.15)$$

where the numbers $\alpha_1, \alpha_2, \dots$ all have the same sign, which is also the sign of α_0 if k is odd, or minus the sign of α_0 if k is even. For convenience we will assume that k is odd. In view of (5.14) we already know that the left hand side of (5.15) is uniformly bounded in n for each fixed point $x \in \mathbb{R}_+ \setminus \{0\}$. Fix such an x . In view of the above observations we have

$$|\alpha_0| < M, \quad \sum_{i=1,2,3,\dots} |\alpha_i|/\text{dist}(x, y_i) < M,$$

uniformly in n , where dist is the Euclidean distance and we used that $x > 0$, $y_i \leq 0$ and all terms in (5.15) have positive sign. But now for any compact set $\mathcal{K}' \subset \mathbb{C} \setminus \mathbb{R}_-$ there exists $R > 0$ so that

$$R^{-1} < |\text{dist}(x, y_i)/\text{dist}(t, y_i)| < R, \quad \text{for all } t \in \mathcal{K}' \text{ and } y_i \in \mathbb{R}_-.$$

This now easily implies that

$$\left| \tilde{P}^{n_2}(t)/\tilde{P}^{n_1}(t) \right| < M,$$

for a new $M > 0$ and for all $t \in \mathcal{K}'$, uniformly in n . This already shows that the family of ratios (5.15) is normal on $\mathbb{C} \setminus \mathbb{R}_-$. Applying (5.14) two times for appropriate indices, we know that for each $t > 0$ there exists a constant $M' > 0$ such that

$$\left| \tilde{P}^{n_2}(t)/\tilde{P}^{n_1}(t) \right| > M',$$

for all n . This observation and Hurwitz' theorem imply that for any compact set $\mathcal{K}' \subset \mathbb{C} \setminus \mathbb{R}_-$, the functions (5.15) are also uniformly bounded from below on \mathcal{K}' by a positive constant. This proves (5.3) for the ratios $P^{n_2}(t)/P^{n_1}(t)$. Similarly, using Theorem 2.7(b) we obtain

$$M^{-1} < |P_{k,n}(t)/P_{k,n+1}(t)| < M, \quad (5.16)$$

for all t in a compact $\mathcal{K} \subset \mathbb{C} \setminus S_-$, uniformly in n . But now the ratio of polynomials in (5.3) can be written as a product of finitely many ratios of the form (5.16) or $P^{n_2}(t)/P^{n_1}(t)$, or their inverses, recall (4.26). This yields Lemma 5.1. \square

Remark 5.7. The proof of Lemma 5.1 simplifies considerably when $k = 0$ or $k = p$. This is because in the former case we deal with the polynomials $Q_n(x)$, whose zeros are uniformly bounded on S_+ , while in the latter case the numerator and denominator in (5.3) are simply constants.

6 Proof of the Widom-type formula

In this section we prove Theorem 2.14. For ease of exposition let us assume for the moment that the period r is sufficiently large: $r \geq p$. The condition (2.23) implies that H is a tridiagonal block Toeplitz operator

$$H = \begin{pmatrix} B_0 & B_{-1} & & \\ B_1 & B_0 & B_{-1} & \\ & B_1 & B_0 & \ddots \\ & & \ddots & \ddots \end{pmatrix}, \quad (6.1)$$

where the blocks B_k are of size $r \times r$ and given by

$$B_0 = \begin{pmatrix} b_0^{(0)} & 1 & & & 0 \\ \vdots & \ddots & \ddots & & \\ b_0^{(p)} & & \ddots & \ddots & \\ & \ddots & & \ddots & 1 \\ 0 & & b_{r-p-1}^{(p)} & \dots & b_{r-1}^{(0)} \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0 & \dots & b_{r-p}^{(p)} & \dots & b_{r-1}^{(1)} \\ \vdots & \ddots & & \ddots & \vdots \\ \vdots & & \ddots & & b_{r-1}^{(p)} \\ \vdots & & & \ddots & \vdots \\ 0 & \dots & \dots & \dots & 0 \end{pmatrix}, \quad (6.2)$$

$$B_{-1} = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ 1 & \mathbf{0} \end{pmatrix}_{r \times r}, \quad (6.3)$$

where $\mathbf{0}$ denotes either a row or a column vector and the 0 in the top right corner is a square matrix of size $r - 1$. The symbol $F(z, x)$ of the block Toeplitz matrix (6.1) is defined as [6, 30]

$$F(z, x) = B_{-1}z^{-1} + B_0 + B_1z - xI_r,$$

where I_r is the identity matrix of size r . One checks that this definition coincides with (2.24). In fact, a similar reasoning can be used also if $r < p$ [7, Sec. 4].

The determinant of a banded block Toeplitz matrix is given by the next result.

Lemma 6.1. (*Widom's determinant identity:*) *Under the assumptions of Theorem 2.14, we have for all n sufficiently large that*

$$Q_{rn}(x) := \det(xI_{rn} - H_{rn}) = \sum_{k=0}^p C_k(x)(z_k(x))^{-n-1}, \quad (6.4)$$

with

$$C_k(x) = \det \left(\frac{1}{2\pi i} \int_{\sigma} F(z, x)^{-1} \frac{dz}{z} \right), \quad (6.5)$$

where $F(z, x)$ is the symbol (2.24) and σ is an arbitrary, clockwise oriented, closed Jordan curve enclosing $z = 0$ and the point $z_k(x)$, but none of the other points $z_j(x)$, $j \in [0 : p]$, $j \neq k$.

Lemma. 6.1 follows by specializing Widom's result [30, Theorem 6.2] to the present setting. We now find a more explicit form for the coefficients $C_k(x)$.

Lemma 6.2. *Under the assumptions of Lemma. 6.1, we have*

$$C_k(x) = \frac{(-1)^r}{f_p} \frac{\det F^{r-1,0}(z_k(x), x)}{\prod_{j \neq k} (z_k(x) - z_j(x))}, \quad k \in [0 : p]. \quad (6.6)$$

Proof. We start from formula (6.5). Note that this formula involves the matrix

$$\tilde{F}(z, x) := F(z, x)^{-1},$$

which can be written in entrywise form as $\tilde{F}(z, x) = (\tilde{F}_{i,j}(z, x))_{i,j=0}^{r-1}$ with

$$\tilde{F}_{i,j}(z, x) = (-1)^{i+j} \frac{\det F^{j,i}(z, x)}{\det F(z, x)}, \quad i, j \in [0 : r-1], \quad (6.7)$$

thanks to the well-known cofactor formula for the inverse of a matrix.

Now we consider in more detail the numerator and denominator of (6.7). For the denominator we have

$$\det F(z, x) \equiv f(z, x) = (-1)^{r-1} z^{-1} + O(1), \quad z \rightarrow 0,$$

by virtue of (1.11)–(1.13) and (2.24). For the numerator we have

$$(\det F^{i,j}(z, x))_{i,j=0}^{r-1} = \frac{(-1)^r}{z} \begin{pmatrix} \mathbf{0} & R \\ 0 & \mathbf{0} \end{pmatrix} + O(1), \quad z \rightarrow 0,$$

where each $\mathbf{0}$ is a row or column vector and where R is an upper triangular matrix with 1's on the diagonal. Indeed, this follows due to the particular form of (2.24), (1.12). Observe that in the matrix $F(z, x)$, the entries with index (i, j) with $j \geq i + 2$ are of order $O(z)$ as $z \rightarrow 0$. This explains the upper triangularity of R . Secondly, the presence of 1's in the first super-diagonal of $F(z, x)$ and of $1/z$ in the bottom left corner of the same matrix implies that for $j = i + 1$, $\det F^{i,j}(z, x) = (-1)^r/z + O(1)$ as $z \rightarrow 0$.

Inserting the above expressions in (6.7), we obtain

$$\tilde{F}(z, x) = \begin{pmatrix} \mathbf{0} & \tilde{R} \\ 0 & \mathbf{0} \end{pmatrix}^T + O(z), \quad z \rightarrow 0, \quad (6.8)$$

for a new upper triangular matrix \tilde{R} with 1's on the diagonal, where T denotes the transpose.

We also need the behavior of $\tilde{F}(z, x)$ for $z \rightarrow z_k(x)$. Note that

$$f(z, x) = \frac{f_p}{z} \prod_{t=0}^p (z - z_t(x)),$$

see (1.13)–(1.14). Hence from (6.7) we have

$$\tilde{F}_{i,j}(z, x) = \frac{(-1)^{i+j}}{f_p} \frac{z}{z - z_k(x)} \frac{\det F^{j,i}(z, x)}{\prod_{t \neq k} (z - z_t(x))}, \quad (6.9)$$

for $i, j \in [0 : r - 1]$. The factor $z - z_k(x)$ in the denominator of (6.9) shows that $\tilde{F}_{i,j}(z, x)$ can have a simple pole at $z = z_k(x)$. Widom [30, Sec. 6] observed that the matrix with the residues,

$$\left(\text{Res}_{z=z_k(x)} \tilde{F}_{i,j}(z, x) \right)_{i,j=0}^{r-1}, \quad (6.10)$$

is a rank-one matrix.

Now we can finish the proof of the lemma. With the contour σ as in the statement of Lemma 6.1, we find from (6.8) and the residue theorem that

$$\frac{1}{2\pi i} \int_{\sigma} \tilde{F}(z, x) \frac{dz}{z} = - \begin{pmatrix} \mathbf{0} & \tilde{R} \\ 0 & \mathbf{0} \end{pmatrix}^T - \frac{1}{z_k(x)} \left(\text{Res}_{z=z_k(x)} \tilde{F}_{i,j}(z, x) \right)_{i,j=0}^{r-1}.$$

Since (6.10) is a rank-one matrix, simple linear algebra then shows that

$$\det \left(\frac{1}{2\pi i} \int_{\sigma} \tilde{F}(z, x) \frac{dz}{z} \right) = - \frac{1}{z_k(x)} \text{Res}_{z=z_k(x)} \tilde{F}_{0,r-1}(z, x).$$

Using (6.9), we conclude that

$$\det \left(\frac{1}{2\pi i} \int_{\sigma} \tilde{F}(z, x) \frac{dz}{z} \right) = \frac{(-1)^r}{f_p} \frac{\det F^{r-1,0}(z_k(x), x)}{\prod_{j \neq k} (z_k(x) - z_j(x))},$$

Comparing this with (6.5), we obtain the desired formula (6.6). \square

Proof of Theorem 2.14. For ease of reference in the next section, we will give the proof for the case of a two-diagonal Hessenberg matrix (2.6). It will be clear that the same proof also works for a general Hessenberg matrix (2.12). From (2.8) we have

$$\begin{pmatrix} 0 & \dots & 0 & a_{rn-1-p} & \dots & -x & 1 \\ \vdots & & \vdots & & \ddots & \ddots & \\ 0 & & 0 & & a_{rn-1} & \dots & -x & 1 \\ \vdots & & \vdots & & \ddots & \ddots & \ddots & \\ 0 & \dots & 0 & & a_{rn+r-2-p} & \dots & -x & 1 \end{pmatrix} \begin{pmatrix} Q_{rn-r}(x) \\ \vdots \\ Q_{rn-1}(x) \\ \vdots \\ Q_{rn+r-1}(x) \end{pmatrix} = \mathbf{0}, \quad (6.11)$$

where the matrix multiplying the column vector, which we call $M(x)$, is of size $r \times 2r$. Let us denote by $B_n(x)$ and $C_n(x)$ the matrices formed by the first r columns and last r columns of $M(x)$, respectively, i.e.

$$B_n(x) = \left(\begin{array}{c|ccc} & a_{rn-p-1} & & -x \\ 0 & & \ddots & \\ \hline 0 & & & a_{rn-1} \\ & & 0 & \end{array} \right), \quad C_n(x) = \begin{pmatrix} 1 & & & & \\ -x & \ddots & & & \\ & \ddots & \ddots & & \\ a_{rn} & & \ddots & \ddots & \\ & \ddots & & \ddots & \ddots \\ & & a_{rn+r-p-2} & & -x & 1 \end{pmatrix}, \quad (6.12)$$

where 0 denotes zero blocks of appropriate sizes, such that the last $r-p-1$ rows and the first $r-p-1$ columns of $B_n(x)$ are zero. Here we are assuming that $r \geq p+1$; the case $r \leq p$ will be discussed in Remark 6.4. Using the vectorial notation

$$\mathbf{Q}_n(x) := (Q_{rn}(x), \dots, Q_{rn+r-1}(x))^T, \quad (6.13)$$

we then write the recurrence (6.11) as

$$\mathbf{Q}_n(x) = A_n(x) \mathbf{Q}_{n-1}(x), \quad \text{with } A_n(x) := -C_n^{-1}(x) B_n(x), \quad (6.14)$$

for $n \geq 1$. Now the periodicity assumption $a_{rn+j} \equiv b_j$ implies that $B_n(x) =: B(x)$, $C_n(x) =: C(x)$ and $A_n(x) =: A(x)$ are all independent of n . By repeatedly using (6.14), this yields

$$\mathbf{Q}_n(x) = A(x)^n \mathbf{Q}_0(x). \quad (6.15)$$

Assume that λ is a non-zero eigenvalue of $A(x)$. Then $\det(B(x) + \lambda C(x)) = 0$. But now

$$B + \lambda C = \begin{pmatrix} \lambda & & & b_{r-p-1} & & -x \\ -\lambda x & \ddots & & & \ddots & \\ & \ddots & \ddots & & & b_{r-1} \\ \lambda b_0 & & \ddots & \ddots & & \\ & \ddots & & \ddots & \ddots & \\ & & \lambda b_{r-p-2} & & -\lambda x & \lambda \end{pmatrix}.$$

If we perform the following operations to $B(x) + \lambda C(x)$: divide rows 2 to r by λ , move row 1 to the bottom and move rows 2 to r one level up, the resulting matrix is exactly $F(1/\lambda, x)$.

Therefore $\det F(1/\lambda, x) = 0$ and $\lambda = 1/z_k(x)$ for some $k \in [0 : p]$. In conclusion, the non-zero eigenvalues of $A(x)$ are given by $1/z_k(x)$, $k \in [0 : p]$. Since the first $r - p - 1$ columns of $A(x)$ are zero, 0 is also an eigenvalue of $A(x)$, with multiplicity $r - p - 1$.

The eigenspace of $A(x)$ associated with the eigenvalue $1/z_k(x)$ is one-dimensional and coincides with the nullspace of $F(z_k(x), x)$. By Cramer's rule it is easy to see that this subspace is spanned by the vector

$$\mathbf{v}_k(x) = \left(\det F^{r-1,0}(z_k(x), x), -\det F^{r-1,1}(z_k(x), x), \dots, (-1)^{r-1} \det F^{r-1,r-1}(z_k(x), x) \right)^T, \quad (6.16)$$

for any $k \in [0 : p]$, whenever the vector (6.16) is nonzero.

Let us show that the first component of $\mathbf{v}_k(x)$ is zero for only finitely many x . Let \mathcal{R} be the compact Riemann surface associated to the algebraic equation $f(z, x) = 0$ whose roots are the functions $z_k(x)$. The collection of functions $\det F^{r-1,0}(z_k(x), x)$, $k \in [0 : p]$, can be seen as a single meromorphic function defined on \mathcal{R} . (It has poles at infinity, see also [29]). Now [7, Lemma 5.5] (see also [7, Lemma 5.6]) shows that \mathcal{R} is connected. Hence the above meromorphic function cannot be identically zero since this would imply by (6.4) and (6.6) that $Q_{rn} \equiv 0$, clearly contradictory. Hence, each function $\det F^{r-1,0}(z_k(x), x)$ has only finitely many zeros in \mathbb{C} .

Now define the matrices

$$\begin{aligned} D(x) &:= \text{diag}(z_0(x)^{-1}, z_1(x)^{-1}, \dots, z_p(x)^{-1}, 0, \dots, 0)_{r \times r}, \\ V(x) &:= (\mathbf{v}_0(x), \mathbf{v}_1(x), \dots, \mathbf{v}_p(x), \mathbf{e}_1, \dots, \mathbf{e}_{r-p-1})_{r \times r}, \end{aligned}$$

where \mathbf{e}_i denotes the standard column unit vector of index i . Then $A(x) = V(x)D(x)V^{-1}(x)$, and so (6.15) gives

$$\mathbf{Q}_n(x) = V(x)D(x)^n V^{-1}(x) \mathbf{Q}_0(x). \quad (6.17)$$

We already know the expression of Q_{rn} , see (6.4) and (6.6). This allows us to find that the first $p + 1$ components of the vector $V^{-1}(x) \mathbf{Q}_0(x)$ are

$$\frac{(-1)^r}{f_p} \frac{1}{\prod_{i \neq 0} (z_0(x) - z_i(x))} z_0(x)^{-1}, \dots, \frac{(-1)^r}{f_p} \frac{1}{\prod_{i \neq p} (z_p(x) - z_i(x))} z_p(x)^{-1}.$$

From this observation and (6.17), the desired formula (2.25) follows immediately.

We have actually shown that (2.25) is valid for all points $x \in \mathbb{C}$ satisfying two conditions, namely that the roots $z_k(x)$, $k \in [0 : p]$ are pairwise distinct, and the vectors $\mathbf{v}_k(x)$, $k \in [0 : p]$, are all nonzero. The collection of points in \mathbb{C} for which the first condition holds but the second fails is finite, as we have already seen. By continuity it is clear that formula (2.25) is also valid for the exceptional points in this finite set.

With (2.25) at our disposal, we can prove as before that the functions $\det F^{r-1,j}(z_k(x), x)$ are zero for only a finite set of $x \in \mathbb{C}$. Finally, to see that the same holds for each function $\det F^{i,j}(z_k(x), x)$, apply formula (2.25) for the monic polynomials associated to the cyclically permuted symbol $Z^{-i-1}F(z, x)Z^{i+1}$. \square

Remark 6.3. Let $x \in \mathbb{C}$ be such that the values $z_k(x)$, $k \in [0 : p]$, are pairwise distinct. We already observed that there are at most finitely many such x with the property that the vector (6.16) is zero. For such x , $\mathbf{v}_k(x)$ will always denote in the next section an eigenvector of $A(x)$ associated with the eigenvalue $1/z_k(x)$.

Remark 6.4. To obtain (6.14) we assumed that $r \geq p + 1$. If $r \leq p$ we proceed as follows. Let $m \in \mathbb{N}$ be large enough so that $\tilde{r} := mr \geq p + 1$. The matrix H , which is periodic of period r , can also be viewed as a periodic matrix of period \tilde{r} . Let $\tilde{F}(z, x)$ be the associated symbol. Linear

algebra shows that the roots $\tilde{z}_k(x)$ of $\det \tilde{F}(z, x) = 0$ are given by $\tilde{z}_k(x) := z_k(x)^m$, $k \in [0 : p]$. Moreover, the null space vector $\tilde{\mathbf{v}}_k(x)$ such that $\tilde{F}(\tilde{z}_k(x), x)\tilde{\mathbf{v}}_k(x) = \mathbf{0}$ can be constructed as follows. With \mathbf{v}_k denoting the vector of length r in (6.16), we define the vector $\tilde{\mathbf{v}}_k$ of length \tilde{r} by

$$\tilde{\mathbf{v}}_k(x) = (\mathbf{v}_k(x)^T, z_k(x)^{-1}\mathbf{v}_k(x)^T, \dots, z_k(x)^{-m+1}\mathbf{v}_k(x)^T)^T. \quad (6.18)$$

With this vector (6.18) playing the role that was played before by $\mathbf{v}_k(x)$, the above proof goes through in exactly the same way as before. This leads again to the same formula (2.25).

7 Ratio and weak asymptotics of Riemann-Hilbert minors

7.1 Generalized Poincaré theorem

The following result is contained in [21], see also [27]. It is closely related to the theory of Krylov subspaces and subspace iteration in numerical linear algebra.

Lemma 7.1. (*Generalized Poincaré theorem.*) Assume that $(A_n)_{n=1}^\infty, A$ are nonsingular matrices of size $m \times m$, $m \in \mathbb{N}$, and $A = \lim_{n \rightarrow \infty} A_n$. Suppose that A is diagonalizable with eigenvalues $\{\lambda_i\}_{i=1}^m$ satisfying

$$|\lambda_1| > |\lambda_2| > \dots > |\lambda_m| > 0.$$

Let $\mathbf{v}_1, \dots, \mathbf{v}_m$ be eigenvectors associated to the eigenvalues $\lambda_1, \dots, \lambda_m$, respectively. Let $(\mathbf{u}_n)_{n=0}^\infty$ be a sequence of column vectors with $\mathbf{u}_0 \neq 0$, generated by the recurrence

$$\mathbf{u}_n = A_n \mathbf{u}_{n-1}, \quad n \geq 1.$$

Then there exists a sequence of complex numbers $(c_n)_n$ such that $c_n \mathbf{u}_n \rightarrow \mathbf{v}_j$, for some $j \in [1 : m]$.

We need a multi-column version of Lemma 7.1.

Lemma 7.2. Under the same assumptions of Lemma 7.1, let $(U_n)_{n=0}^\infty$ be a sequence of matrices of size $m \times l$, $l \in [1 : m]$, with U_0 having linearly independent columns, such that

$$U_n = A_n U_{n-1}, \quad n \geq 1.$$

Then there exists a sequence $(C_n)_{n=0}^\infty$ of invertible, upper triangular matrices of size $l \times l$ such that

$$\lim_{n \rightarrow \infty} U_n C_n = (\mathbf{v}_{j_1}, \mathbf{v}_{j_2}, \dots, \mathbf{v}_{j_l}),$$

the matrix with columns $\mathbf{v}_{j_1}, \dots, \mathbf{v}_{j_l}$, where j_1, \dots, j_l are l distinct indices in $[1 : m]$. Here the limit is defined entrywise.

Proof. We prove this lemma by induction on l . For $l = 1$ it reduces to Lemma 7.1. Let us assume as induction hypothesis that the result holds for the index $l - 1$. Thus there exists a sequence of upper triangular, invertible matrices C_n of size $l - 1$ such that, if we write

$$M_n := (\mathbf{u}_n^{(1)}, \dots, \mathbf{u}_n^{(l-1)}) C_n, \quad (7.1)$$

with $\mathbf{u}_n^{(i)}$ denoting the i th column of U_n , then

$$\lim_{n \rightarrow \infty} M_n = (\mathbf{v}_{j_1}, \mathbf{v}_{j_2}, \dots, \mathbf{v}_{j_{l-1}}), \quad (7.2)$$

where j_1, \dots, j_{l-1} are distinct indices in $[1 : m]$.

For n large enough, there exists a unique column vector \mathbf{d}_n such that the vector

$$\mathbf{w}_n := \mathbf{u}_n^{(l)} + M_n \mathbf{d}_n \in \mathbb{C}^m \quad (7.3)$$

does not have a contribution from the vectors $\mathbf{v}_{j_1}, \mathbf{v}_{j_2}, \dots, \mathbf{v}_{j_{l-1}}$, i.e., if we write \mathbf{w}_n in terms of the basis $\{\mathbf{v}_i\}_{i=1}^m$ of \mathbb{C}^m , then the coefficients multiplying $\mathbf{v}_{j_1}, \dots, \mathbf{v}_{j_{l-1}}$ are zero. To see this, observe that finding the coefficients of \mathbf{d}_n amounts to solve a non-homogeneous linear system whose coefficient matrix tends to the identity matrix, thanks to (7.2). Observe that $\mathbf{w}_n \neq \mathbf{0}$ for all n , because otherwise we would have a linear dependency between the columns of U_n and therefore (by the recursion $U_n = A_n U_{n-1}$ with A_n nonsingular) between the columns of U_0 , contrary to the assumptions of the lemma.

From (7.1), (7.3) and the recursion $U_n = A_n U_{n-1}$ we have

$$\begin{aligned} A_n \mathbf{w}_{n-1} &= \mathbf{u}_n^{(l)} + (\mathbf{u}_n^{(1)}, \dots, \mathbf{u}_n^{(l-1)}) C_{n-1} \mathbf{d}_{n-1} \\ &= \mathbf{u}_n^{(l)} + M_n C_{n-1}^{-1} C_{n-1} \mathbf{d}_{n-1} \\ &= \mathbf{w}_n + M_n (C_{n-1}^{-1} C_{n-1} \mathbf{d}_{n-1} - \mathbf{d}_n) \\ &=: \mathbf{w}_n + M_n \mathbf{f}_n. \end{aligned} \quad (7.4)$$

Note that for n sufficiently large, \mathbf{f}_n is the unique column vector for which $A_n \mathbf{w}_{n-1} - M_n \mathbf{f}_n$ has no contribution from the vectors $\mathbf{v}_{j_1}, \mathbf{v}_{j_2}, \dots, \mathbf{v}_{j_{l-1}}$. Equivalently, since the $\{\mathbf{v}_i\}_{i=1}^m$ are eigenvectors for A , \mathbf{f}_n is the unique column vector for which $(A_n - A) \mathbf{w}_{n-1} - M_n \mathbf{f}_n$ has no contribution from the vectors $\mathbf{v}_{j_1}, \mathbf{v}_{j_2}, \dots, \mathbf{v}_{j_{l-1}}$. This yields the estimate

$$|\mathbf{f}_n| \leq c \|A_n - A\| |\mathbf{w}_{n-1}| \quad (7.5)$$

for a suitable constant c and for all n sufficiently large, on account of (7.2). Here we write $|\cdot|$ for the Euclidean norm of a vector and $\|\cdot\|$ for the induced matrix norm.

Define

$$B_n := A_n - M_n \frac{\mathbf{f}_n \mathbf{w}_{n-1}^H}{\mathbf{w}_{n-1}^H \mathbf{w}_{n-1}},$$

with \mathbf{w}_{n-1}^H denoting the conjugate transpose of \mathbf{w}_{n-1} . From (7.4) we get

$$B_n \mathbf{w}_{n-1} = \mathbf{w}_n$$

while (7.2), (7.5) and the fact that $A_n \rightarrow A$ imply that

$$\|B_n - A_n\| \rightarrow 0, \quad n \rightarrow \infty.$$

We can now apply Lemma 7.1 to the matrices $(B_n)_n$ and the vectors $(\mathbf{w}_n)_n$. This yields a sequence of nonzero constants $(c_n)_n$ such that $c_n \mathbf{w}_n \rightarrow \mathbf{v}_{j_l}$ for a certain index $j_l \in [1 : m]$. By the very construction of \mathbf{w}_n we have that $j_l \notin \{j_1, \dots, j_{l-1}\}$. The definition of the new sequence of upper triangular matrices of size $l \times l$ is obvious. \square

7.2 Ratio and weak asymptotics of Riemann-Hilbert minors

We will apply Lemma 7.2 to the polynomials Q_n generated by the three-term recurrence (1.1),

$$xQ_n(x) = Q_{n+1}(x) + a_{n-p}Q_{n-p}(x), \quad n \geq p. \quad (7.6)$$

We will assume that the recurrence coefficients a_n are asymptotically periodic with period $r \in \mathbb{N}$, where we may assume without loss of generality that $r \geq p + 1$. The case $r \leq p$ can be handled by enlarging the period and/or by using Remark 6.4.

We write the recurrence relation in matrix-vector form as (6.14), recalling (6.12)–(6.13). Since the recurrence coefficients a_n are asymptotically periodic with period r , we have

$$\lim_{n \rightarrow \infty} (A_n(x), B_n(x), C_n(x)) = (A(x), B(x), C(x)), \quad (7.7)$$

with $A(x) := -C(x)^{-1}B(x)$ and $B(x), C(x)$ as in (6.12) but with a_{rn+j} replaced by b_j , $j \in [0 : r-1]$, and where the limits of the matrices are taken entrywise.

In the proof of Theorem 2.14 we observed that the matrix $A(x)$ is closely related to the block Toeplitz symbol $F(z, x)$. More precisely, we showed that the non-zero eigenvalues of $A(x)$ are the inverted roots $1/z_k(x)$, $k \in [0 : p]$ and the corresponding eigenvectors $\mathbf{v}_k(x)$ are given by (6.16) (see also Remark 6.3). In addition the matrix $A(x)$ has zero as an eigenvalue of multiplicity $r - p - 1$.

The second kind functions $\Psi_n^{(k)}$ are defined in (1.5). They satisfy the same recurrence relation (7.6) for all $n \geq p$. In analogy with (6.13) we set

$$\Psi_n^{(k)}(x) := \left(\Psi_{rn}^{(k)}(x), \dots, \Psi_{rn+r-1}^{(k)}(x) \right)^T \quad (7.8)$$

and we define the RH-type matrix

$$U_n(x) := \begin{pmatrix} \mathbf{Q}_n(x) & \Psi_n^{(1)}(x) & \dots & \Psi_n^{(p)}(x) \end{pmatrix}. \quad (7.9)$$

Then the recurrence (6.14) extends to

$$U_n(x) = A_n(x)U_{n-1}(x), \quad n \geq 1. \quad (7.10)$$

Note the similarity with Lemma 7.2. Hence the next result should not come as a surprise.

Proposition 7.3. *Let $U_n(x)$ be the RH type matrix in (7.9). For any fixed $x \in \mathbb{C} \setminus \bigcup_j \Gamma_j$, there exists a sequence of invertible upper triangular matrices $C_n(x)$ of size $p+1$ such that*

$$\lim_{n \rightarrow \infty} \begin{pmatrix} U_{n-1}(x) \\ U_n(x) \end{pmatrix} C_n(x) = \begin{pmatrix} \mathbf{v}_{j_0}(x) & \dots & \mathbf{v}_{j_p}(x) \\ z_{j_0}^{-1}(x)\mathbf{v}_{j_0}(x) & \dots & z_{j_p}^{-1}(x)\mathbf{v}_{j_p}(x) \end{pmatrix} \quad (7.11)$$

where (j_0, \dots, j_p) is a permutation of $[0 : p]$ (depending on x), and with $\mathbf{v}_k(x)$ defined in (6.16), see also Remark 6.3.

Proof. Throughout the proof we will drop the x -dependence for convenience. We want to apply the generalized Poincaré theorem (Lemma 7.2) to the recurrence (7.10). Recall that only the last $p+1$ columns of A_n are nonzero. So the matrix A_n could have zero as an eigenvalue (of multiplicity $r - p - 1$), contrary to the assumptions of Lemma 7.2. To resolve this issue, we partition

$$U_n =: \begin{pmatrix} \tilde{U}_n \\ \hat{U}_n \end{pmatrix}, \quad \mathbf{v}_j =: \begin{pmatrix} \tilde{\mathbf{v}}_j \\ \hat{\mathbf{v}}_j \end{pmatrix},$$

with \tilde{U}_n and \hat{U}_n having $r - p - 1$ and $p + 1$ rows respectively, and similarly for $\tilde{\mathbf{v}}_j$ and $\hat{\mathbf{v}}_j$. We also partition

$$A_n =: \begin{pmatrix} 0 & \tilde{A}_n \\ 0 & \hat{A}_n \end{pmatrix}, \quad A =: \begin{pmatrix} 0 & \tilde{A} \\ 0 & \hat{A} \end{pmatrix},$$

with \hat{A}_n and \hat{A} square matrices of size $p + 1$.

Recall that the nonzero eigenvalues of A are z_j^{-1} , $j \in [0 : p]$, and the corresponding eigenvectors are \mathbf{v}_j . With the above partitions, this yields

$$\mathbf{0} = (A - z_j^{-1}I)\mathbf{v}_j = \begin{pmatrix} -z_j^{-1}I & \tilde{A} \\ 0 & \hat{A} - z_j^{-1}I \end{pmatrix} \begin{pmatrix} \tilde{\mathbf{v}}_j \\ \hat{\mathbf{v}}_j \end{pmatrix}, \quad (7.12)$$

for all $j \in [0 : p]$. In particular, the matrix \hat{A} is diagonalizable with z_j^{-1} as eigenvalues, $j \in [0 : p]$, and the corresponding eigenvectors are $\hat{\mathbf{v}}_j$ (note also that (7.12) implies $\hat{\mathbf{v}}_j \neq \mathbf{0}$).

The recursion (7.10) becomes

$$\begin{pmatrix} \tilde{U}_n \\ \hat{U}_n \end{pmatrix} = \begin{pmatrix} 0 & \tilde{A}_n \\ 0 & \hat{A}_n \end{pmatrix} \begin{pmatrix} \tilde{U}_{n-1} \\ \hat{U}_{n-1} \end{pmatrix}. \quad (7.13)$$

In particular,

$$\hat{U}_n = \hat{A}_n \hat{U}_{n-1}, \quad n \geq 1. \quad (7.14)$$

We can now apply Lemma 7.2 to the matrices $(\hat{A}_n)_n$ and $(\hat{U}_n)_n$. Observe that the matrices \hat{U}_n are all nonsingular (and so the matrices \hat{A}_n); in fact, $\det \hat{U}_n(x)$ is a nonzero constant (independent of x), as it follows from Prop. 2.6. Lemma 7.2 yields a sequence of invertible upper triangular matrices C_n such that

$$\hat{U}_{n-1}C_n \rightarrow (\hat{\mathbf{v}}_{j_0} \quad \dots \quad \hat{\mathbf{v}}_{j_p}) \quad (7.15)$$

as $n \rightarrow \infty$, for a certain permutation (j_0, \dots, j_p) of $[0 : p]$. (Note that we write C_n instead of C_{n-1} .) Applying (7.14) and (7.15) we then get

$$\hat{U}_n C_n = \hat{A}_n \hat{U}_{n-1} C_n \rightarrow \begin{pmatrix} z_{j_0}^{-1} \hat{\mathbf{v}}_{j_0} & \dots & z_{j_p}^{-1} \hat{\mathbf{v}}_{j_p} \end{pmatrix},$$

as $n \rightarrow \infty$, where we used that $\hat{A}_n \rightarrow \hat{A}$ and $\hat{\mathbf{v}}_j$ is an eigenvector of \hat{A} for the eigenvalue z_j^{-1} . On the other hand, from the first block row of (7.13) we have

$$\tilde{U}_n C_n = \tilde{A}_n \hat{U}_{n-1} C_n \rightarrow \tilde{A} (\hat{\mathbf{v}}_{j_0} \quad \dots \quad \hat{\mathbf{v}}_{j_p}) = \begin{pmatrix} z_{j_0}^{-1} \tilde{\mathbf{v}}_{j_0} & \dots & z_{j_p}^{-1} \tilde{\mathbf{v}}_{j_p} \end{pmatrix},$$

as $n \rightarrow \infty$, where the equality follows from the first block row of (7.12). Finally,

$$\tilde{U}_{n-1} C_n = \tilde{A}_{n-1} \hat{U}_{n-2} C_n = \tilde{A}_{n-1} (\hat{A}_{n-1})^{-1} \hat{U}_{n-1} C_n \rightarrow (\tilde{\mathbf{v}}_{j_0} \quad \dots \quad \tilde{\mathbf{v}}_{j_p}).$$

Combining the above limits, the proposition is proved. \square

In principle, the indices j_0, \dots, j_p in Prop. 7.3 could depend on x . We will see further that this is not the case; in fact we have $j_0 = 0$, $j_1 = 1$, and so on.

Fix $k \in [0 : p]$. Taking determinants of suitable $(k+1) \times (k+1)$ minors of (7.11) and using the fact that C_n is upper triangular, we find that

$$\lim_{n \rightarrow \infty} B_{k, rn-1}(x) c_n(x) = (-1)^{k(k+1)/2} \det(\mathbf{v}'_{j_0}(x), \dots, \mathbf{v}'_{j_k}(x)), \quad (7.16)$$

$$\lim_{n \rightarrow \infty} B_{k, rn+r-1}(x) c_n(x) = (-1)^{k(k+1)/2} (z_{j_0}^{-1} \dots z_{j_k}^{-1}) \det(\mathbf{v}'_{j_0}(x), \dots, \mathbf{v}'_{j_k}(x)), \quad (7.17)$$

for any fixed $x \in \mathbb{C} \setminus \bigcup_j \Gamma_j$, where c_n denotes the determinant of the principal $(k+1) \times (k+1)$ submatrix of C_n , and where the vector \mathbf{v}'_j consists of the last $k+1$ entries of \mathbf{v}_j . For convenience we introduce the following notation:

$$S_k := \begin{cases} S_+, & \text{for } k \text{ even, } k \in [0 : p], \\ S_-, & \text{for } k \text{ odd, } k \in [0 : p]. \end{cases} \quad (7.18)$$

Lemma 7.4. *Let $k \in [0 : p]$, and let x be a fixed point in $\mathbb{C} \setminus (\bigcup_j \Gamma_j \cup S_k)$. Then in (7.16)–(7.17) we have*

$$\det(\mathbf{v}'_{j_0}(x), \dots, \mathbf{v}'_{j_k}(x)) \neq 0. \quad (7.19)$$

Proof. Let $\widehat{\mathbf{v}}_j$ consist of the last $p+1$ rows of \mathbf{v}_j , as in the proof of Prop. 7.3. Recall that the columns of the matrix

$$(\widehat{\mathbf{v}}_{j_0}(x), \dots, \widehat{\mathbf{v}}_{j_k}(x)) \quad (7.20)$$

are linearly independent, since they are eigenvectors corresponding to distinct eigenvalues of the matrix \widehat{A} (see the proof of Prop. 7.3). In particular, there exist $k+1$ row indices such that the minor obtained by selecting these rows in (7.20) is nonzero. Denoting the value of this minor with $\kappa(x) \neq 0$ and taking the determinant of the corresponding $(k+1) \times (k+1)$ minor in (7.11), we get

$$\lim_{n \rightarrow \infty} B^{(n_0, n_1, \dots, n_k)}(x) c_n(x) = \pm \kappa(x) \neq 0,$$

for suitable indices n_i with $rn - p - 1 \leq n_0 < n_1 < \dots < n_k \leq rn - 1$, with again c_n the determinant of the principal $(k+1) \times (k+1)$ submatrix of C_n . Comparing this to (7.16), we get

$$\lim_{n \rightarrow \infty} B_{k, rn-1}(x) / B^{(n_0, n_1, \dots, n_k)}(x) = \pm \det(\mathbf{v}'_{j_0}(x), \dots, \mathbf{v}'_{j_k}(x)) / \kappa(x).$$

Now if (7.19) fails, then this limit would be zero, thereby contradicting Lemma 5.1 (see also (3.4)). \square

Remark 7.5. The above proof shows that *any* $(k+1) \times (k+1)$ minor of

$$\begin{pmatrix} \mathbf{v}_{j_0}(x) & \dots & \mathbf{v}_{j_k}(x) \\ z_{j_0}^{-1}(x) \mathbf{v}_{j_0}(x) & \dots & z_{j_k}^{-1}(x) \mathbf{v}_{j_k}(x) \end{pmatrix},$$

obtained by selecting $k+1$ rows with the difference between the smallest and largest row index not exceeding p , is nonzero if $x \in \mathbb{C} \setminus (\bigcup_j \Gamma_j \cup S_k)$. Also recall Remark 6.3.

By Lemma 7.4, we can take the ratio of (7.16) and (7.17) and get the pointwise limit

$$\lim_{n \rightarrow \infty} B_{k, rn-1}(x) / B_{k, rn+r-1}(x) = z_{j_0}(x) \dots z_{j_k}(x), \quad x \in \mathbb{C} \setminus \left(\bigcup_j \Gamma_j \cup S_k \right).$$

Similar arguments can be applied for the other residue classes modulo r , showing that

$$\lim_{n \rightarrow \infty} B_{k, n}(x) / B_{k, n+r}(x) = z_{j_0}(x) \dots z_{j_k}(x), \quad x \in \mathbb{C} \setminus \left(\bigcup_j \Gamma_j \cup S_k \right). \quad (7.21)$$

Proposition 7.6. *In Prop. 7.3 we have for any fixed $x \in \mathbb{C} \setminus (S_+ \cup S_-)$,*

$$(j_0, \dots, j_p) = (0, \dots, p). \quad (7.22)$$

Prop. 7.6 will be proved in Section 7.3. In the latter section we also prove Theorem 2.2, in particular we show that $\Gamma_k \subset S_k$ for all $k \in [0 : p]$. Note that we did not use Theorem 2.2 so far. Lemma 5.1 and (7.21)–(7.22) imply that, uniformly on compact subsets of $\mathbb{C} \setminus S_k$,

$$\lim_{n \rightarrow \infty} B_{k, n}(x) / B_{k, n+r}(x) = z_0(x) \dots z_k(x). \quad (7.23)$$

We are now ready for the

Proof of Theorem 2.1. For any measure μ in \mathbb{C} denote its logarithmic potential $U^\mu(x)$ as

$$U^\mu(x) = - \int \log |x - s| \, d\mu(s). \quad (7.24)$$

We will prove formula (2.2) for each sequence $\mu_{k,n}$ with n of the form $rm + l$, $l \in [0 : r - 1]$ fixed. Denote with κ_n the leading coefficient of the polynomial $B_{k,n}(x)$. We have uniformly for x in compact subsets of $\mathbb{C} \setminus S_k$ that

$$\begin{aligned} \lim_{n \rightarrow \infty} \left(U^{\mu_{k,n}}(x) - \frac{1}{n} \log |\kappa_n| \right) &= - \lim_{n \rightarrow \infty} \frac{1}{n} \log |B_{k,n}(x)| \\ &= - \lim_{m \rightarrow \infty} \frac{1}{rm + l} \sum_{j=1}^m \log |B_{k,rj+l}(x) / B_{k,r(j-1)+l}(x)| = \frac{1}{r} \log \prod_{j=0}^k |z_j(x)| = U^{\mu_k}(x) + c, \end{aligned} \quad (7.25)$$

with c a constant independent of x , and μ_k the measure in (1.19). Here the first equality is obvious from the definitions (7.24) and (2.1), the second one follows by telescopic cancelation, the third one follows by (7.23) and the fourth one by [7, Prop. 5.10]. It is easy to see that $\log |\kappa_n|/n$ is bounded from below as a function of n , due to (5.1) and Prop. 5.5.

Let $C_0(S_k)$ denote the space of continuous functions on the star S_k that vanish at infinity. Since $\|\mu_{k,n}\| \leq (p - k)/p$ for all n (cf. Lemma 1.2), it follows from the Banach-Alaoglu theorem that we can extract a subsequence from $\mu_{k,n}$ that converges in the weak-star topology to a finite measure ν supported on S_k . Let $x_0 \in \mathbb{C} \setminus S_k$ be a fixed point. From the weak-star convergence and (7.25) we deduce that for every $x \in \mathbb{C} \setminus S_k$,

$$\int \log \left| \frac{x_0 - s}{x - s} \right| \, d\nu(s) = U^{\mu_k}(x) - U^{\mu_k}(x_0) = \frac{1}{r} \operatorname{Re} \left(\log \prod_{j=0}^k z_j(x) \right) + \tilde{c}, \quad (7.26)$$

where $\log \prod_{j=0}^k z_j(x)$ is a holomorphic branch of the logarithm of $\prod_{j=0}^k z_j(x)$ and \tilde{c} is some constant. Note that $\phi(s) := \log \left| \frac{x_0 - s}{x - s} \right| \in C_0(S_k)$, so the weak-star convergence indeed applies.

We claim that

$$\int_{S_k} \log(1 + |s|) \, d\nu(s) < \infty. \quad (7.27)$$

This will be justified at the end of the proof and now we complete the argument as follows. From (7.27) we obtain that U^ν is well-defined and superharmonic in \mathbb{C} , and in particular we can replace the first integral in (7.26) by $U^\nu(x) - U^\nu(x_0)$. Note also that the last relation in (7.25), which is in fact valid for all $x \in \mathbb{C}$, implies that U^{μ_k} is continuous everywhere in the complex plane. This in turn implies, using (7.26) and the superharmonicity of U^ν , that U^ν is bounded on every compact segment of S_k . Now we are in a position to apply Theorem II.1.4 from [25], which gives $\mu_k = \nu$.

Now we justify (7.27). This is equivalent to say that $U^\nu(x) > -\infty$ for any fixed $x \in \mathbb{C} \setminus S_k$. It is clear that we can construct a non-increasing sequence of functions $(k_m(y))_{m \in \mathbb{N}}$ in $C_0(S_k)$ satisfying $k_m(y) = \log(1/|x - y|)$ whenever $\log(1/|x - y|) \geq -m$ and $k_m(y) \geq -m$ for all $y \in S_k$. Applying a standard monotone convergence theorem argument to this sequence k_m together with (7.25) and the weak-star convergence to ν , it is easy to deduce that $U^\nu(x) \geq U^{\mu_k}(x) + c$, for some other constant c .

Summarizing, we obtain that (2.2) is valid for every $\phi \in C_0(S_k)$. Since $\|\mu_{k,n}\| \leq (p - k)/p$ and $\|\mu_k\| = (p - k)/p$, the convergence in the weak-star topology of $\mu_{k,n}$ to μ_k implies that the sequence $\mu_{k,n}$ is tight. This implies again by a standard argument that (2.2) is also valid for bounded continuous functions on S_k . \square

Remark 7.7. Due to the interlacing properties described in Theorem 4.6, it is easy to see that the conclusion of Theorem 2.1 remains valid for the zeros of general Riemann-Hilbert minors $P^{(n+i_0, n+i_1, \dots, n+i_k)}$, with $i_j \in \mathbb{Z}$, $j \in [0 : k]$, a fixed set of indices satisfying (5.2).

For later use, we state the next lemma.

Lemma 7.8. *Fix $0 \leq k < l \leq p$. Uniformly for x in compact subsets of $\mathbb{C} \setminus S_k$, we have*

$$\lim_{n \rightarrow \infty} \frac{B_{k,l,rn}(x)}{B_{k,rn}(x)} = \frac{\begin{vmatrix} \tilde{f}_0(z_0(x), x) & \dots & \tilde{f}_0(z_k(x), x) \\ \vdots & & \vdots \\ \tilde{f}_{k-1}(z_0(x), x) & \dots & \tilde{f}_{k-1}(z_k(x), x) \\ \tilde{f}_l(z_0(x), x) & \dots & \tilde{f}_l(z_k(x), x) \end{vmatrix}}{\begin{vmatrix} \tilde{f}_0(z_0(x), x) & \dots & \tilde{f}_0(z_k(x), x) \\ \vdots & & \vdots \\ \tilde{f}_{k-1}(z_0(x), x) & \dots & \tilde{f}_{k-1}(z_k(x), x) \\ \tilde{f}_k(z_0(x), x) & \dots & \tilde{f}_k(z_k(x), x) \end{vmatrix}}, \quad (7.28)$$

where the functions $\tilde{f}_j(z, x)$ are defined in (8.16), and $B_{k,l,n}(x) := B^{(n-l, n-k+1, \dots, n-1, n)}(x)$. Denoting with $\tilde{\mathcal{A}}_k$ the set of $x \in \mathbb{C}$ for which the denominator in (7.28) is zero, then the set $\tilde{\mathcal{A}}_k$ is finite. For any $x \in \tilde{\mathcal{A}}_k \setminus S_k$ the right hand side of (7.28) has a removable pole at x .

Proof. Remark 7.5 shows that the determinants in the numerator and denominator in (7.28) are both nonzero if $x \in \mathbb{C} \setminus (S_+ \cup S_-)$. For any fixed such x , we obtain (7.28) by taking determinants of suitable submatrices in Prop. 7.3 (with $j_i = i$ for all i). Lemma 5.1 shows the convergence holds uniformly on compact subsets of $\mathbb{C} \setminus S_k$.

Finally, let us prove that $\tilde{\mathcal{A}}_k$ is finite. Let \mathcal{R} be the compact Riemann surface with $(k+1)!(\frac{p+1}{k+1})$ sheets which are labeled by the ordered $(k+1)$ -tuples (i_0, i_1, \dots, i_k) in $[0 : p]$. On the sheet (i_0, i_1, \dots, i_k) we cut away all the sets Γ_{i_j} and $\Gamma_{i_{j-1}}$, $j \in [0 : k]$. If we cross such a cut, then we move to the sheet labeled by $(\tilde{i}_0, \tilde{i}_1, \dots, \tilde{i}_k)$ where $\tilde{z}_{\tilde{i}_j}$ is the analytic continuation of z_{i_j} through the cut. Now in the denominator of (7.28) we can replace the role of z_0, \dots, z_k by z_{i_0}, \dots, z_{i_k} . The collection of all these functions yields a meromorphic function on \mathcal{R} . Since we already know that this function is not identically zero on the sheet $(0, 1, \dots, k)$, it can indeed have only finitely many zeros on that sheet. (Note that \mathcal{R} can be disconnected; in that case we restrict ourselves to the connected component(s) involving the sheet $(0, 1, \dots, k)$). \square

7.3 Proofs of Proposition 7.6 and Theorem 2.2

In this section we prove Prop. 7.6 and Theorem 2.2. Note that we did not use Theorem 2.2 prior to the statement of Prop. 7.6. Moreover, it is a general fact [7, Prop. 1.1] that each set Γ_k associated to a Hessenberg matrix H consists of a finite union of analytic arcs.

For each $k \in [0 : p]$, the sequence $(B_{k,n}(x)/B_{k,n+r}(x))_n$ for n tending to infinity converges pointwise for $x \in \mathbb{C} \setminus (\bigcup_j \Gamma_j \cup S_k)$, by (7.21), and it is a normal family in $\mathbb{C} \setminus S_k$ by Lemma 5.1, recall the definition of S_k in (7.18). Therefore, the convergence in (7.21) is in fact uniform on compact subsets of $\mathbb{C} \setminus S_k$ and the limit function $z_{j_0}(x) \dots z_{j_k}(x)$ is analytic there. Applying this observation subsequently for $k \in [0 : p]$, we see that for each $x \in \mathbb{C}$, there exists a permutation $(\tilde{z}_j(x))_{j=0}^p$ of the set $(z_j(x))_{j=0}^p$ so that \tilde{z}_0 is analytic in $\mathbb{C} \setminus S_+$, $\tilde{z}_0 \tilde{z}_1$ is analytic in $\mathbb{C} \setminus S_-$, $\tilde{z}_0 \tilde{z}_1 \tilde{z}_2$ is analytic in $\mathbb{C} \setminus S_+$, $\tilde{z}_0 \tilde{z}_1 \tilde{z}_2 \tilde{z}_3$ is analytic in $\mathbb{C} \setminus S_-$, and so on (alternatingly with S_+ and S_-). In fact $\tilde{z}_i := z_{j_i}$, $i \in [0 : p]$. We also deduce from (1.16) that as $x \rightarrow \infty$, $\tilde{z}_0(x) = x^{-r} + O(x^{-r-1})$ and $\tilde{z}_j(x) = O(x^{r/p})$, $j \in [1 : p]$, since it is clear that for x sufficiently large, $\tilde{z}_0(x) = z_0(x)$.

7.3.1 Proofs of Prop. 7.6 and Theorem 2.2(a)

The proof will proceed in a very similar way to the one in [8, Section 4]. For convenience, we will list the main highlights of the proof but we will sometimes refer to [8] for the details. We will assume without loss of generality that r is a multiple of $p+1$, see also Remark 7.10 below.

We already observed that for each $k \in [0 : p]$,

$$\lim_{n \rightarrow \infty} B_{k,n}(x)/B_{k,n+r}(x) = \pm \lim_{n \rightarrow \infty} P_{k,n}(x)/P_{k,n+r}(x) = \tilde{z}_0(x) \dots \tilde{z}_k(x), \quad x \in \mathbb{C} \setminus S_k. \quad (7.29)$$

It follows from (7.29) and Theorem 2.7(a) that the functions \tilde{z}_i satisfy the symmetry property $\tilde{z}_i(\omega x) = \tilde{z}_i(x)$, where $\omega = \exp(2\pi i/(p+1))$; recall that r is a multiple of $p+1$. In accordance to this symmetry property, we define the functions

$$\tilde{y}_k(x) = \tilde{z}_k(x^{1/(p+1)}), \quad k \in [0 : p],$$

where we take the principal branch of $x^{1/(p+1)}$. (By the symmetry property, the choice of the branch is irrelevant.) Note that $\tilde{y}_k(x)$ is analytic in $\mathbb{C} \setminus \mathbb{R}$.

Let us introduce the notation

$$\mathbb{R}_k := (-1)^k \mathbb{R}_+, \quad k \in [0 : p].$$

We now define a measure s_k on \mathbb{R}_k with density

$$ds_k(x) = \frac{1}{2\pi i} \frac{p+1}{r} \left(\frac{\tilde{y}'_{k,+}(x)}{\tilde{y}_{k,+}(x)} - \frac{\tilde{y}'_{k,-}(x)}{\tilde{y}_{k,-}(x)} \right) dx, \quad x \in \mathbb{R}_k, \quad (7.30)$$

$k \in [0 : p-1]$, where the prime denotes the derivative with respect to x , and where the $+$ and $-$ subscripts stand for the boundary values obtained from the upper or lower half of the complex plane, respectively. Note that the measure (7.30) is well-defined except for finitely many x , and its density is integrable near each endpoint of its support [11]. We claim that s_k is a real-valued (possibly signed) measure on \mathbb{R}_k with total mass

$$s_k(\mathbb{R}_k) := \int_{\mathbb{R}_k} ds_k(x) = \frac{p-k}{p}, \quad k \in [0 : p-1]. \quad (7.31)$$

Indeed, since the polynomials $P_{k,n}$ (2.9) have real coefficients, it follows from (7.29) that $\tilde{z}_i(\bar{x}) = \overline{\tilde{z}_i(x)}$, where the bar denotes complex conjugation. This shows that s_k is a real-valued measure. For $x \in \mathbb{C} \setminus \mathbb{R}_k$, we have

$$\int_{\mathbb{R}_k} \frac{ds_k(t)}{x-t} = \frac{1}{2\pi i} \frac{p+1}{r} \sum_{j=0}^k \int_{\mathbb{R}_k} \frac{1}{x-t} \left(\frac{\tilde{y}'_{j,+}(t)}{\tilde{y}_{j,+}(t)} - \frac{\tilde{y}'_{j,-}(t)}{\tilde{y}_{j,-}(t)} \right) dt = -\frac{p+1}{r} \sum_{j=0}^k \frac{\tilde{y}'_j(x)}{\tilde{y}_j(x)}, \quad (7.32)$$

where in the first equality we used the fact that $\sum_{j=0}^{k-1} \tilde{y}'_j(x)/\tilde{y}_j(x) = (\log \prod_{j=0}^{k-1} \tilde{y}_j(x))'$ is analytic across \mathbb{R}_k , and the second equality follows by contour deformation and the residue theorem. From the behavior of the functions $\tilde{y}_j(x)$ near infinity we see that the right hand side of (7.32) behaves as $\frac{p-k}{p}x^{-1} + o(x^{-1})$ as $x \rightarrow \infty$. This implies (7.31).

We then obtain from (7.30)–(7.31) that

$$\frac{1}{\pi} \frac{p+1}{r} \int_{\mathbb{R}_k} \operatorname{Im} \left(\frac{\tilde{y}'_{k,+}(x)}{\tilde{y}_{k,+}(x)} \right) dx = \frac{p-k}{p}, \quad k \in [0 : p-1], \quad (7.33)$$

with Im denoting the imaginary part of a complex number.

As in [8], we now turn to the construction of a second collection of auxiliary measures. The functions $z_k(x)$ are unambiguously defined in the complement of $\bigcup_{k=0}^{p-1} \Gamma_k$, which is a finite union of analytic arcs. Consider the functions

$$y_k(x) := z_k(x^{1/(p+1)}), \quad k \in [0 : p],$$

where we take the principal branch of $x^{1/(p+1)}$. The \pm -boundary values of y_k and y'_k are well-defined at almost every point $x \in \tilde{\Gamma}_k := \Gamma_k^{p+1}$. This allows us to introduce the measures

$$d\sigma_k(x) := \frac{1}{2\pi i} \frac{p+1}{r} \sum_{j=0}^k \left(\frac{y'_{j,+}(x)}{y_{j,+}(x)} - \frac{y'_{j,-}(x)}{y_{j,-}(x)} \right) dx, \quad x \in \tilde{\Gamma}_k. \quad (7.34)$$

The measure σ_k is closely related to the measure μ_k in (1.19). In fact, for any Borel set B ,

$$\sigma_k(B) = \mu_k(h^{-1}(B)) \quad (7.35)$$

where h is the map $x \mapsto x^{p+1}$. In particular, σ_k is a positive measure and

$$\sigma_k(\tilde{\Gamma}_k) = \frac{p-k}{p}, \quad k \in [0 : p-1]. \quad (7.36)$$

Alternatively, (7.36) could be proved directly by using the same argument as in (7.31).

Now let us take a fixed open interval $J \subset \mathbb{R}$ that does not contain any intersection points or endpoints of the analytic arcs constituting $\tilde{\Gamma}_k$, for every k . We also ask J not to contain isolated intersection points of the sets $\tilde{\Gamma}_k$ with the real axis. Thus there exists an open connected set $U \subset \mathbb{C}$ such that $U \cap \mathbb{R} = J$ and moreover $U \cap \tilde{\Gamma}_k$ is either empty or equal to J , for any $k \in [0 : p-1]$. The boundary values $y_{k,+}(x)$ for $x \in J$ are then uniquely defined and they vary analytically with x .

On the interval J , there exist indices $0 \leq m_1 < m_2 < \dots < m_L < p$ such that

$$|y_{0,+}(x)| = \dots = |y_{m_1,+}(x)| < |y_{m_1+1,+}(x)| = \dots = |y_{m_2,+}(x)| < \dots < |y_{m_L+1,+}(x)| = \dots = |y_{p,+}(x)|, \quad (7.37)$$

for all $x \in J$. We define $m_0 := -1$ and $m_{L+1} := p$.

We will see later that $m_{k+1} - m_k \in \{1, 2\}$ for all k , i.e., each “cluster” $|y_{m_k+1,+}(x)| = \dots = |y_{m_{k+1},+}(x)|$ in (7.37) can only have length 1 or 2. The Cauchy-Riemann equations imply [8]

$$\operatorname{Im} \left(\frac{y'_{m_k+1,+}(x)}{y_{m_k+1,+}(x)} \right) \geq \dots \geq \operatorname{Im} \left(\frac{y'_{m_{k+1},+}(x)}{y_{m_{k+1},+}(x)} \right), \quad x \in J, \quad k \in [0 : L], \quad (7.38)$$

and the numbers in (7.38) satisfy the pairing

$$\operatorname{Im} \left(\frac{y'_{m_k+j,+}(x)}{y_{m_k+j,+}(x)} \right) = -\operatorname{Im} \left(\frac{y'_{m_{k+1}+1-j,+}(x)}{y_{m_{k+1}+1-j,+}(x)} \right), \quad j = 1, \dots, m_{k+1} - m_k. \quad (7.39)$$

The underlying reason for (7.39) is that for any $x \in \mathbb{R}$, the numbers $y_{k,+}(x)$, $k \in [0 : p]$, are either real or they come in complex conjugate pairs. This is trivial if $x > 0$. If $x < 0$ it can be seen e.g. with the help of Lemma 1.3, taking into account that r is a multiple of $p+1$.

Now we get

$$\begin{aligned} \sum_{k=0}^{p-1} \frac{p-k}{p} &\geq \sum_{k=0}^{p-1} \sigma_k(\mathbb{R}) \geq \frac{1}{2\pi} \frac{p+1}{r} \int_{\mathbb{R}} \sum_{k=0}^p \left| \operatorname{Im} \left(\frac{y'_{k,+}(x)}{y_{k,+}(x)} \right) \right| dx \\ &= \frac{1}{2\pi} \frac{p+1}{r} \int_{\mathbb{R}} \sum_{k=0}^p \left| \operatorname{Im} \left(\frac{\tilde{y}'_{k,+}(x)}{\tilde{y}_{k,+}(x)} \right) \right| dx \\ &= \frac{1}{\pi} \frac{p+1}{r} \sum_{k=0}^{p-1} \int_{\mathbb{R}_k} \left| \operatorname{Im} \left(\frac{\tilde{y}'_{k,+}(x)}{\tilde{y}_{k,+}(x)} \right) \right| dx \geq \sum_{k=0}^{p-1} \frac{p-k}{p}, \end{aligned}$$

where the first relation uses (7.36) and the positivity of σ_k , the second relation follows exactly like in [8, Sec. 4], the third one follows since the numbers \tilde{y}_k form a permutation of the y_k , and the fifth relation is a consequence of (7.33). Finally, the fourth relation uses that on \mathbb{R}_+ we have $\operatorname{Im} \left(\frac{\tilde{y}'_{2k,+}(x)}{\tilde{y}_{2k,+}(x)} \right) = -\operatorname{Im} \left(\frac{\tilde{y}'_{2k+1,+}(x)}{\tilde{y}_{2k+1,+}(x)} \right)$ and on \mathbb{R}_- we have $\operatorname{Im} \left(\frac{\tilde{y}'_{2k,+}(x)}{\tilde{y}_{2k,+}(x)} \right) = -\operatorname{Im} \left(\frac{\tilde{y}'_{2k-1,+}(x)}{\tilde{y}_{2k-1,+}(x)} \right)$; for \mathbb{R}_+ this follows since

$$\sum_{j=2k}^{2k+1} \operatorname{Im} \left(\frac{\tilde{y}'_{j,+}(x)}{\tilde{y}_{j,+}(x)} \right) = \frac{1}{2i} \sum_{j=2k}^{2k+1} \left(\frac{\tilde{y}'_{j,+}(x)}{\tilde{y}_{j,+}(x)} - \frac{\tilde{y}'_{j,-}(x)}{\tilde{y}_{j,-}(x)} \right) = 0, \quad x \in \mathbb{R}_+,$$

due to the fact that $(\log \tilde{y}_{2k} \tilde{y}_{2k+1})'$ is analytic on \mathbb{R}_+ .

From the above chain of inequalities we obtain:

Lemma 7.9. (a) *We have*

$$\tilde{\Gamma}_k \subset \mathbb{R}, \quad k \in [0 : p-1]. \quad (7.40)$$

(b) *Clusters of length ≥ 3 in (7.37) cannot occur.*

The proof is exactly as in [8].

From Lemma 7.9(a)–(b) we see that $\tilde{\Gamma}_k \cap \tilde{\Gamma}_{k-1}$ contains at most finitely many points. We also have that $(\log y_0 \dots y_{k-1})'$ is analytic in $\mathbb{C} \setminus \tilde{\Gamma}_{k-1}$, so in particular this holds on the interior of each interval of $\tilde{\Gamma}_k$. Then (7.34)–(7.36) imply that

$$\frac{1}{\pi} \frac{p+1}{r} \int_{\tilde{\Gamma}_k} \operatorname{Im} \left(\frac{y'_{k,+}(x)}{y_{k,+}(x)} \right) dx = \sigma_k(\tilde{\Gamma}_k) = \frac{p-k}{p}, \quad k \in [0 : p-1]. \quad (7.41)$$

The measure σ_k is non-trivial on each subarc of $\tilde{\Gamma}_k$ (see the proof of Lemma 7.9(a)). From the positivity of σ_k we also have

$$\operatorname{Im} \left(\frac{y'_{k,+}(x)}{y_{k,+}(x)} \right) \begin{cases} > 0, & x \in \operatorname{int}(\tilde{\Gamma}_k), \\ = -\operatorname{Im} \left(\frac{y'_{k-1,+}(x)}{y_{k-1,+}(x)} \right) \leq 0, & x \in \tilde{\Gamma}_{k-1}, \\ = 0, & x \in \mathbb{R} \setminus (\tilde{\Gamma}_k \cup \tilde{\Gamma}_{k-1}), \end{cases} \quad (7.42)$$

where $\operatorname{int}(\tilde{\Gamma}_k)$ denotes the interior of $\tilde{\Gamma}_k$ in the topology of \mathbb{R} , where the first equality uses (7.39) and Lemma 7.9(b).

Recall that $\tilde{y}_k(x)$ is analytic for $x \in \mathbb{C} \setminus \mathbb{R}$. By Lemma 7.9(a) the same holds for the function $y_k(x)$. Thus for each fixed $k \in [0 : p]$ we have that $\tilde{y}_k(x) = y_{j_k}(x)$ for all $x \in \mathbb{C} \setminus \mathbb{R}$ and for a certain j_k which is independent of x . From (7.41)–(7.42) and (7.33) we now easily find by induction on $k = 0, 1, \dots$ that $j_k = k$ and moreover $\tilde{\Gamma}_k \subset \mathbb{R}_k$. This proves Proposition 7.6 and Theorem 2.2(a). \square

Remark 7.10. In the above proof we assumed that r is a multiple of $p+1$. For general r , the symmetry properties take the form $\tilde{z}_k(\omega x) = \omega^{-r} \tilde{z}_k(x)$ and $z_k(\omega x) = \omega^{-r} z_k(x)$ (Recall Lemma 1.3). Then the functions \tilde{y}_k and y_k have a jump on the whole of \mathbb{R}_- . But the logarithmic derivatives do not have such a jump, therefore the proof goes through in exactly the same way as above.

7.3.2 Proof of Theorem 2.2(b)–(c)

Fix $k \in [0 : p-1]$ and let r be arbitrary. Let I be an interval of $\tilde{\Gamma}_k \subset \mathbb{R}_k$. We will assume that $\tilde{\Gamma}_k$ is not the whole set \mathbb{R}_+ or \mathbb{R}_- since otherwise there is nothing to prove. We claim that

$$\frac{r}{p+1}\sigma_k(I) = \begin{cases} \in \mathbb{N}, & \text{if } I \cap \{0, \infty\} = \emptyset, \\ \in \mathbb{N}/(p+1), & \text{if } 0 \in I, \\ \in \mathbb{N}/p, & \text{if } \infty \in I. \end{cases} \quad (7.43)$$

Let us assume this for the moment. By breaking each interval I of $\tilde{\Gamma}_k$ in smaller subintervals if necessary, in such a way that (7.43) remains valid, we may assume that the left hand side of (7.43) always lies in the range $(0, 1]$. Then we have from the total mass of σ_k in (7.36) that

$$a + \frac{b}{p+1} + \frac{c}{p} = \frac{r}{p+1} \frac{p-k}{p} = \frac{k+1}{p+1} r - \frac{kr}{p}, \quad (7.44)$$

where a denotes the number of intervals I of $\tilde{\Gamma}_k$ for which the left hand side of (7.43) equals 1, and where $b \in [0 : p]$ and $c \in [0 : p-1]$ are nonzero *only if* there is an interval $I \subset \tilde{\Gamma}_k$ containing 0 or ∞ respectively and with the left hand side of (7.43) being < 1 . Since p and $p+1$ are coprime, from (7.44) we deduce that

$$\frac{b}{p+1} = \frac{k+1}{p+1} r - \left\lfloor \frac{k+1}{p+1} r \right\rfloor, \quad \text{and} \quad \frac{c}{p} = \left\lceil \frac{kr}{p} \right\rceil - \frac{kr}{p}.$$

Inserting this in (7.44) we get

$$a = \left\lfloor \frac{k+1}{p+1} r \right\rfloor - \left\lceil \frac{kr}{p} \right\rceil.$$

We then find for the total number n_k of intervals of $\tilde{\Gamma}_k$ that

$$n_k = \left\lfloor \frac{k+1}{p+1} r \right\rfloor - \left\lceil \frac{kr}{p} \right\rceil + \mathbf{1}_{b \neq 0} + \mathbf{1}_{c \neq 0} = \left\lfloor \frac{k+1}{p+1} r \right\rfloor - \left\lceil \frac{kr}{p} \right\rceil,$$

where the indicator function $\mathbf{1}_{x \neq 0}$ equals 1 if $x \neq 0$ and zero otherwise, and where the second equality uses that $b \neq 0$ if and only if $(k+1)r/(p+1) \notin \mathbb{N}$ and similarly $c \neq 0$ if and only if $kr/p \notin \mathbb{N} \cup \{0\}$. This proves Theorem 2.2(b)–(c).

Finally we prove (7.43). Due to (7.35) it will be enough to prove that

$$r\mu_k(J) = \begin{cases} \in \mathbb{N}, & \text{if } J \cap \{\infty\} = \emptyset, \\ \in \mathbb{N}/p, & \text{if } \infty \in J, \end{cases} \quad (7.45)$$

for any connected component J of $\Gamma_k \subset S_k$. Thus J is either a line segment on $S_k \setminus \{0\}$ (there are $p+1$ rotations of such a segment), or it is a set of the form $J = \{x \in S_k \mid |x| \leq a\}$ for some $a > 0$.

The first statement of (7.45) follows from [7, Prop 2.10]. Let us check it directly if $J \subset S_k$ is a line segment of the form $[a, b]$ with $a, b \notin \{0, \infty\}$. From the definition (1.19) of μ_k it is easy to see that

$$r\mu_k(J) = \frac{1}{2\pi} \lim_{x \rightarrow b, x \in J} \left(\arg \prod_{j=0}^k z_{j,+}(x) - \arg \prod_{j=0}^k z_{j,-}(x) \right) \quad (7.46)$$

where we take the argument function \arg so that $\arg \prod_{j=0}^k z_j(x)$ is continuous in $U \setminus J$ with U a complex neighborhood of $[a, b]$ (U excludes b). This is possible since $\prod_{j=0}^k z_j(x)$ is analytic and

nonzero in $\mathbb{C} \setminus \Gamma_k$. But then the expression between brackets in (7.46) is an integral multiple of 2π , yielding the first statement in (7.45).

To prove the second statement of (7.45), note that (7.46) remains valid if b lies at ∞ . In that case, the behavior of the functions $z_k(x)$ near infinity in (1.17) implies that the expression between brackets in (7.46) is an integral multiple of $2\pi/p$. This proves (7.45). \square

8 Nikishin system

In this section we prove Theorem 2.10 on the connection with Nikishin systems. We start by recalling some ideas in [2].

8.1 Multiple orthogonality relations

For any $l \in [0 : p]$, we define the sequence of monic polynomials $(Q_{n,l}(x))_{n=l}^{\infty}$ by the recurrence relation

$$xQ_{n,l}(x) = Q_{n+1,l}(x) + a_{n-p} Q_{n-p,l}(x), \quad n \geq l, \quad (8.1)$$

with initial conditions

$$Q_{l,l}(x) \equiv 1, \quad Q_{l-1,l}(x) \equiv \cdots \equiv Q_{l-p,l}(x) \equiv 0. \quad (8.2)$$

Note that $\deg Q_{n,l} = n-l$ and that $Q_{n,0}(x) \equiv Q_n(x)$. Moreover, the $p+1$ sequences $(Q_{n,l}(x))_{n=0}^{\infty}$ form a basis for the space of all solutions $(q_n)_{n=0}^{\infty}$ to the difference equation

$$xq_n = q_{n+1} + a_{n-p} q_{n-p}, \quad n \geq p.$$

Lemma 8.1. *(The measures ν_1, \dots, ν_p ; see [2]:) Suppose that $a_n > 0$ for all n and the numbers a_n are uniformly bounded. There exists an increasing sequence of positive integers $(n_j)_{j=0}^{\infty}$ such that for any fixed $l \in [1 : p]$, we have*

(a)

$$\lim_{j \rightarrow \infty} \frac{Q_{(p+1)n_j,l}(x)}{Q_{(p+1)n_j}(x)} = \int \frac{d\nu_l(t)}{x-t}, \quad (8.3)$$

uniformly for x in compact subsets of $\mathbb{C} \setminus S_+$, where ν_l is a compactly supported measure on S_+ .

(b) *The moments of ν_l are uniquely determined from the condition (8.3), independently of the choice of the sequence $(n_j)_{j=0}^{\infty}$.*

(c) *The measure ν_l can be written as*

$$d\nu_l(t) = t^{1-l} d\tilde{\nu}_l(t^{p+1}), \quad (8.4)$$

for a compactly supported, positive measure $\tilde{\nu}_l$ supported on $S_+^{p+1} = \mathbb{R}_+$. Thus for $l = 1$ the measure ν_l is rotationally invariant under rotations over $2\pi/(p+1)$ while for $l > 1$ it is rotationally invariant up to a monomial factor.

Lemma 8.1 was shown by Aptekarev-Kalyagin-Van Iseghem [2]. The key fact for (8.3) is that for any $n \in \mathbb{N}$ and $l \in [1 : p]$ the zeros of $Q_{(p+1)n,l}(x)$ and $Q_{(p+1)n}(x)$ interlace (in a suitable sense) on the star-like set S_+ . The existence of a sequence $(n_j)_{j=0}^{\infty}$ such that (8.3) holds then follows from the Helly selection theorem, see [2].

Recall from Theorem 1.1 that the polynomials $Q_n(x)$ are multiple orthogonal with respect to the measures ν_1, \dots, ν_p defined in (8.3), in the sense of (1.4).

We will assume throughout this section that we are in the exactly periodic case (2.17) and that (2.18) holds. We can assume that the sequence n_j in (8.3) is such that each $(p+1)n_j$ is a multiple of r ; this follows directly from the freedom in choosing a convergent subsequence in the proof of Lemma 8.1 in [2]. For a fixed $l \in [0 : p]$, let

$$T_n(x) := Q_{n+l,l}(x), \quad n \geq 0.$$

By (8.1) and (8.2), this sequence satisfies

$$xT_n(x) = T_{n+1}(x) + a_{n+l-p}T_{n-p}(x), \quad n \geq 0, \quad (8.5)$$

with initial conditions

$$T_0(x) \equiv 1, \quad T_{-1}(x) \equiv \dots \equiv T_{-p}(x) \equiv 0. \quad (8.6)$$

It follows that the block Toeplitz symbol associated with $(T_n)_{n=0}^\infty$ is $Z^{-l}F(z, x)Z^l$, with Z and F given by (1.12) and (1.10), respectively.

Using Theorem 2.14 and simple considerations, we deduce that for any index sequence $(n_j)_{j=0}^\infty$ as described above, we have

$$\lim_{j \rightarrow \infty} \frac{Q_{(p+1)n_j, l}(x)}{Q_{(p+1)n_j}(x)} = \lim_{j \rightarrow \infty} \frac{T_{(p+1)n_j - l}(x)}{Q_{(p+1)n_j}(x)} = \frac{f_l(z_0(x), x)}{f_0(z_0(x), x)}, \quad l \in [0 : p], \quad (8.7)$$

uniformly on compact subsets of $\mathbb{C} \setminus S_+$, where the functions f_l are the following minors of the block Toeplitz symbol:

$$\begin{aligned} f_0(z, x) &= (-1)^r z^{-1} \det F^{r-1, 0}(z, x), \\ f_l(z, x) &= (-1)^l \det F^{l-1, 0}(z, x), \quad l \in [1 : r]. \end{aligned} \quad (8.8)$$

Note that the functions $f_l(z_0(x), x)$, $l \in [0 : p]$, are analytic in $\mathbb{C} \setminus \Gamma_0$, and that $f_0(z, x)$ has an extra factor z^{-1} in comparison to the other functions $f_l(z, x)$. Let

$$\mathcal{A}_0 := \{x \in \mathbb{C} \setminus \Gamma_0 : \frac{f_l(z_0(x), x)}{f_0(z_0(x), x)} \text{ has a non-removable pole at } x \text{ for some } l \in [1 : p]\}.$$

From the statement of Theorem 2.14 we know that the set \mathcal{A}_0 is finite. Moreover, since the functions $Q_{n,l}(x)/Q_n(x)$ are analytic on $\mathbb{C} \setminus S_+$, we deduce from (8.7) that $\mathcal{A}_0 \subset S_+$.

8.2 Formal Nikishin system

In this section we will introduce a hierarchy of functions $f_{l,k}$, $0 \leq k < l \leq p$, which will be identified later as the Cauchy transforms of certain measures that form the different layers of a Nikishin system on $(\Gamma_0, \dots, \Gamma_{p-1})$.

From (8.3) and (8.7) we obtain

$$\int \frac{d\nu_l(t)}{x-t} = \frac{f_l(z_0(x), x)}{f_0(z_0(x), x)} =: f_{l,0}(z_0(x), x), \quad l \in [1 : p], \quad (8.9)$$

for $x \in \mathbb{C} \setminus (\Gamma_0 \cup \mathcal{A}_0)$. The measures ν_l and the functions $f_{l,0}$ will form layer 0 of the Nikishin hierarchy, as we will show later in this section. We also deduce that the measures ν_l are supported on $\Gamma_0 \cup \mathcal{A}_0$.

We consider the functions

$$\frac{f_{l,0}(z_{0,+}(x), x) - f_{l,0}(z_{0,-}(x), x)}{f_{1,0}(z_{0,+}(x), x) - f_{1,0}(z_{0,-}(x), x)}, \quad x \in \Gamma_0,$$

$l \in [2 : p]$. (These expressions will be well-defined as we will show in the next section.) The relations $z_{0,\pm} = z_{1,\mp}$ on Γ_0 imply that these functions can be meromorphically extended to $\mathbb{C} \setminus \Gamma_1$; we denote the resulting functions by

$$f_{l,1}(z_0(x), z_1(x), x) := \frac{f_{l,0}(z_1(x), x) - f_{l,0}(z_0(x), x)}{f_{1,0}(z_1(x), x) - f_{1,0}(z_0(x), x)}, \quad (8.10)$$

and we observe that $f_{l,1}$ is a symmetric function of its two arguments z_0 and z_1 . This will form layer 1 of the Nikishin hierarchy.

Next we consider the functions

$$\frac{f_{l,1}(z_0(x), z_{1,+}(x), x) - f_{l,1}(z_0(x), z_{1,-}(x), x)}{f_{2,1}(z_0(x), z_{1,+}(x), x) - f_{2,1}(z_0(x), z_{1,-}(x), x)}, \quad x \in \Gamma_1,$$

$l \in [3 : p]$. They can be extended to $\mathbb{C} \setminus \Gamma_2$ by the functions

$$f_{l,2}(z_0(x), z_1(x), z_2(x), x) := \frac{f_{l,1}(z_0(x), z_2(x), x) - f_{l,1}(z_0(x), z_1(x), x)}{f_{2,1}(z_0(x), z_2(x), x) - f_{2,1}(z_0(x), z_1(x), x)}, \quad (8.11)$$

and we observe that $f_{l,2}$ is a symmetric function of its three arguments z_0, z_1, z_2 (this is a bit harder to see now). This will form layer 2 of the Nikishin hierarchy.

We can continue this procedure and set

$$\begin{aligned} & f_{l,k}(z_0(x), \dots, z_k(x), x) \\ &= \frac{f_{l,k-1}(z_0(x), \dots, z_{k-2}(x), z_k(x), x) - f_{l,k-1}(z_0(x), \dots, z_{k-2}(x), z_{k-1}(x), x)}{f_{k,k-1}(z_0(x), \dots, z_{k-2}(x), z_k(x), x) - f_{k,k-1}(z_0(x), \dots, z_{k-2}(x), z_{k-1}(x), x)}, \end{aligned}$$

for $l \in [k+1 : p]$ and $k \in [1 : p-1]$, using induction on k . It allows a determinantal formula:

Lemma 8.2. *Consider the functions f_l , $l \in [0 : p]$ in (8.8). Define the hierarchy of functions $f_{l,k}$, $0 \leq k < l \leq p$, as explained above. Abbreviating $f_l(z_k(x)) := f_l(z_k(x), x)$ for each l , we have*

$$f_{l,k}(z_0(x), \dots, z_k(x), x) = \frac{\begin{vmatrix} f_0(z_0(x)) & \dots & f_0(z_k(x)) \\ \vdots & & \vdots \\ f_{k-1}(z_0(x)) & \dots & f_{k-1}(z_k(x)) \\ f_l(z_0(x)) & \dots & f_l(z_k(x)) \end{vmatrix}}{\begin{vmatrix} f_0(z_0(x)) & \dots & f_0(z_k(x)) \\ \vdots & & \vdots \\ f_{k-1}(z_0(x)) & \dots & f_{k-1}(z_k(x)) \\ f_k(z_0(x)) & \dots & f_k(z_k(x)) \end{vmatrix}}. \quad (8.12)$$

We also have

$$\begin{aligned} & f_{l,k,+}(z_0(x), \dots, z_k(x), x) - f_{l,k,-}(z_0(x), \dots, z_k(x), x) = \\ & \frac{\begin{vmatrix} f_0(z_0(x)) & \dots & f_0(z_{k-1}(x)) \\ \vdots & & \vdots \\ f_{k-1}(z_0(x)) & \dots & f_{k-1}(z_{k-1}(x)) \end{vmatrix} \begin{vmatrix} f_0(z_0(x)) & \dots & f_0(z_{k+1}(x)) \\ \vdots & & \vdots \\ f_k(z_0(x)) & \dots & f_k(z_{k+1}(x)) \\ f_l(z_0(x)) & \dots & f_l(z_{k+1}(x)) \end{vmatrix}}{\begin{vmatrix} f_0(z_0(x)) & \dots & f_0(z_k(x)) \\ \vdots & & \vdots \\ f_k(z_0(x)) & \dots & f_k(z_k(x)) \end{vmatrix} \begin{vmatrix} f_0(z_0(x)) & \dots & f_0(z_{k-1}(x)) & f_0(z_{k+1}(x)) \\ \vdots & & \vdots & \vdots \\ f_k(z_0(x)) & \dots & f_k(z_{k-1}(x)) & f_k(z_{k+1}(x)) \end{vmatrix}}, \quad (8.13) \end{aligned}$$

for $x \in \Gamma_k$, where in the right hand side of (8.13) we define the values $z_j(x)$ as the limiting values obtained from the $+$ -side of Γ_k (picking another labeling of the $z_j(x)$ so that (1.15) holds can only change the sign in (8.13)), and where we set the determinant of an empty matrix as 1.

Proof. The determinantal formulas follow by induction on $k = 0, 1, 2, \dots$ by means of a basic linear algebra calculation using Sylvester's determinant identity [14]. See also [1, Sec. 8]. \square

Remark 8.3. As in the proof of Lemma 7.8, we see that the denominator in the right-hand side of (8.12) vanishes only for finitely many $x \in \mathbb{C}$. Taking into account the relations $z_{i,\pm} = z_{i+1,\mp}$ on Γ_i , $i \in [0 : k-1]$, we see that the ratio in (8.12) is in fact analytic in $\mathbb{C} \setminus (\Gamma_k \cup \mathcal{A}_k)$, where \mathcal{A}_k is a finite set in $\mathbb{C} \setminus \Gamma_k$.

Lemma 8.4. *Let r be a multiple of p and assume the ordering (2.18). For any $0 \leq k < l \leq p$ there exists $C \neq 0$ such that the following asymptotics hold for $x \rightarrow \infty$:*

$$\left| \begin{array}{ccc} f_0(z_0(x)) & \dots & f_0(z_k(x)) \\ \vdots & & \vdots \\ f_{k-1}(z_0(x)) & \dots & f_{k-1}(z_k(x)) \\ f_l(z_0(x)) & \dots & f_l(z_k(x)) \end{array} \right| / \left| \begin{array}{ccc} f_0(z_0(x)) & \dots & f_0(z_k(x)) \\ \vdots & & \vdots \\ f_{k-1}(z_0(x)) & \dots & f_{k-1}(z_k(x)) \\ f_k(z_0(x)) & \dots & f_k(z_k(x)) \end{array} \right| = Cx^{k-l}(1 + O(x^{-p-1})).$$

The proof of Lemma 8.4 is postponed to Section 8.4.

For convenience, we define a new symbol:

$$\widehat{F}(z, x) := P_r F(z, x)^T P_r, \quad (8.14)$$

where P_r is the $r \times r$ permutation matrix that consists of 1's in the main antidiagonal and 0's elsewhere, i.e., the (i, j) entry of P_r equals $\delta_{i+j-r+1}$, for $i, j \in [0 : r-1]$. Note that $\widehat{F}(z, x)$ is the reflection of $F(z, x)$ with respect to its main antidiagonal. We can rewrite (8.8) as

$$\begin{aligned} f_0(z, x) &= (-1)^r z^{-1} \det \widehat{F}^{r-1,0}(z, x), \\ f_l(z, x) &= (-1)^l \det \widehat{F}^{r-1,r-l}(z, x), \quad l \in [1 : r]. \end{aligned} \quad (8.15)$$

We also define the functions

$$\begin{aligned} \tilde{f}_0(z, x) &= (-1)^r z^{-1} \det F^{r-1,0}(z, x), \\ \tilde{f}_l(z, x) &= (-1)^l \det F^{r-1,r-l}(z, x), \quad l \in [1 : r]. \end{aligned} \quad (8.16)$$

8.3 Proof of Theorem 2.10

Lemma 8.2 asserts that the measures ν_j form a *formal* Nikishin system in the sense of [1]. To prove Theorem 2.10, we will now show that they are a *true* Nikishin system.

Theorem 8.5. (*Nikishin property.*) *Let H be the two-diagonal Hessenberg matrix (2.6), with entries $a_n > 0$ that satisfy (2.17)–(2.18). For each pair of indices k, l with $0 \leq k < l \leq p$,*

$$f_{l,k}(z_0(x), \dots, z_k(x), x) = \int \frac{d\nu_{l,k}(t)}{x - t}, \quad (8.17)$$

for a measure $\nu_{l,k}$ supported on $\Gamma_k \cup \mathcal{A}_k$, where \mathcal{A}_k is a finite subset of $S_k \setminus \Gamma_k$ ($S_- \setminus \Gamma_k$) if k is even (odd). The measure $\nu_{l,k}$ takes the form (2.20), for a measure $\tilde{\nu}_{l,k}$ with constant sign supported on \mathbb{R}_+ (\mathbb{R}_-) if k is even (odd).

Proof. By (8.9) we know that (8.17) is valid for $k = 0$ and $l \in [1 : p]$. We also know already that the measures $\nu_{l,0}$ are supported on $\Gamma_0 \cup \mathcal{A}_0$, with \mathcal{A}_0 a finite subset of $S_+ \setminus \Gamma_0$, and that (2.20) holds for $k = 0$.

In what follows we are going to work with the polynomials associated with the symbol \widehat{F} introduced in (8.14). That is, we consider now the two-diagonal Hessenberg operator \widehat{H} with periodic structure whose first r coefficients are given in the following order:

$$a_{r-p-1}, a_{r-p-2}, \dots, a_1, a_0, a_{r-1}, \dots, a_{r-p}.$$

We associate with the new operator \widehat{H} the polynomials $P_{k,l,n}$ as defined in Section 2.4. These are the polynomials we will employ below.

Let $1 \leq k < l \leq p$. Applying Theorem 2.12,

$$\frac{P_{k,l,n}(x)}{P_{k,n}(x)} = x^{k-l} \frac{\widetilde{P}_{k,l,n}(x^{p+1})}{\widetilde{P}_{k,n}(x^{p+1})},$$

where the zeros of $\widetilde{P}_{k,n}$ and $\widetilde{P}_{k,l,n}$ lie in \mathbb{R}_+ (\mathbb{R}_-) if k is even (odd), and are weakly interlacing.

Let us denote by d_n the degree of $\widetilde{P}_{k,n}$. Thanks to the weak interlacing property, we know that either $\deg \widetilde{P}_{k,l,n} = d_n$ or $\deg \widetilde{P}_{k,l,n} = d_n + 1$. In any case, we can write

$$\frac{\widetilde{P}_{k,l,n}(z)}{z \widetilde{P}_{k,n}(z)} = \alpha_{-1,n} + \sum_{i=0}^{d_n} \frac{\alpha_{i,n}}{z - x_{i,n}},$$

where $x_{0,n} := 0$, $\{x_{i,n}\}_{i=1}^{d_n}$ denotes the zeros of $\widetilde{P}_{k,n}$, and for $i \geq 0$, we set $\alpha_{i,n} = 0$ if $z - x_{i,n}$ is a common factor of the numerator and denominator. It also follows from the interlacing property that all the coefficients $\{\alpha_{i,n}\}_{i=0}^{d_n}$ have the same sign.

Assume for the moment that k is even, so the zeros of $\widetilde{P}_{k,n}$ lie in \mathbb{R}_+ . Let $\nu_{l,k,n}$ be the discrete measure supported on $\{x_{i,n}\}_{i=0}^{d_n}$ with mass $\alpha_{i,n}$ at $x_{i,n}$. Hence

$$\frac{\widetilde{P}_{k,l,n}(z)}{z \widetilde{P}_{k,n}(z)} = \alpha_{-1,n} + \int \frac{d\nu_{l,k,n}(t)}{z - t}. \quad (8.18)$$

It is easy to check that $\alpha_{-1,n} \leq 0$ if $\nu_{l,k,n} \geq 0$, and $\alpha_{-1,n} \geq 0$ if $\nu_{l,k,n} \leq 0$. Therefore the function (8.18) maps $(-\infty, 0)$ into $(-\infty, 0)$ if $\nu_{l,k,n}$ is positive, and maps $(-\infty, 0)$ into $(0, \infty)$ if $\nu_{l,k,n}$ is negative. Moreover, it maps the upper half plane into the lower half plane (upper half plane) if $\nu_{l,k,n}$ is positive (negative).

An important ingredient in our proof is formula (7.28), which certainly applies in our situation. We should apply this formula for the B -polynomials associated with the operator \widehat{H} (or the symbol \widehat{F}). Therefore, taking into account (8.15)–(8.16), the determinants in the right-hand side of (7.28) are in this situation constructed with the functions $f_k(z, x)$.

We know by Lemma 7.8 that

$$\lim_{n \rightarrow \infty} \frac{\widetilde{P}_{k,l,n}(z)}{z \widetilde{P}_{k,n}(z)} =: G(z),$$

uniformly on compact subsets of $\mathbb{C} \setminus [0, \infty)$. Since $G \not\equiv 0$, the measures $\nu_{l,k,n}$ are all positive for n sufficiently large, or they are all negative for n sufficiently large. Therefore G is an analytic function in $\mathbb{C} \setminus [0, \infty)$ that satisfies one of the following properties:

- (a) $G(x) < 0$ for all $x \in (-\infty, 0)$ and G maps the upper half plane into the lower half plane,
- (b) $G(x) > 0$ for all $x \in (-\infty, 0)$ and G maps the upper half plane into the upper half plane.

Applying Theorem A.4 from [18], we deduce that

$$G(z) = \alpha + \int \frac{d\hat{\nu}_{l,k}(t)}{z-t}, \quad z \in \mathbb{C} \setminus [0, \infty),$$

where $\alpha \in \mathbb{R}$ and $\hat{\nu}_{l,k}$ is a measure with constant sign supported on \mathbb{R}_+ .

In particular, applying the above equations together with (7.28), (3.4), and (8.12), we obtain

$$\lim_{n \rightarrow \infty} \frac{P_{k,l,n}(x)}{P_{k,n}(x)} = x^{k-l+p+1} \left(\alpha + \int \frac{d\hat{\nu}_{l,k}(t)}{x^{p+1}-t} \right) = (-1)^{l-k} f_{l,k}(z_0(x), \dots, z_k(x), x), \quad (8.19)$$

uniformly on compact subsets of $\mathbb{C} \setminus S_+$. Now, Lemma 8.4 implies that $\alpha = 0$. Finally, using

$$\frac{x^{k-l+p+1}}{x^{p+1}-t} = \frac{1}{p+1} \sum_{m=0}^p \frac{(e^{\frac{2\pi i m}{p+1}} s)^{k-l+1}}{x - e^{\frac{2\pi i m}{p+1}} s}, \quad t = s^{p+1},$$

we deduce

$$x^{k-l+p+1} \int_{\mathbb{R}_+} \frac{d\hat{\nu}_{l,k}(t)}{x^{p+1}-t} = \frac{1}{p+1} \int_{S_+} \frac{s^{k-l+1}}{x-s} d\hat{\nu}_{l,k}(s^{p+1}).$$

This justifies (8.17) and (2.20) with $\tilde{\nu}_{l,k} := \frac{(-1)^{l-k}}{p+1} \hat{\nu}_{l,k}$. and we see from (8.19) that the function $f_{l,k}(z_0(x), \dots, z_k(x), x)$ has no singularities outside S_+ . The proof is analogous for odd values of k . It is clear from the analyticity of $f_{l,k}(z_0(x), \dots, z_k(x), x)$ on $\mathbb{C} \setminus (\Gamma_k \cup \mathcal{A}_k)$, see Remark 8.3, that the measure $\nu_{l,k}$ is supported on $\Gamma_k \cup \mathcal{A}_k$. \square

Proof of Theorem 2.10: We have precisely shown in Theorem 8.5 that if $d\nu_{l,k}(x) = g_{l,k}(x) dx + d\nu_{l,k}^{(s)}(x)$ denotes the Lebesgue decomposition of $\nu_{l,k}$, then for $l \in [k+2 : p]$,

$$\frac{g_{l,k}(x)}{g_{k+1,k}(x)} = \frac{f_{l,k,+}(z_0(x), \dots, z_k(x), x) - f_{l,k,-}(z_0(x), \dots, z_k(x), x)}{f_{k+1,k,+}(z_0(x), \dots, z_k(x), x) - f_{k+1,k,-}(z_0(x), \dots, z_k(x), x)}, \quad x \in \Gamma_k,$$

is expressible as the Cauchy transform of a measure $\nu_{l,k+1}$ supported on the star complementary to Γ_k . \square

8.4 Proof of Lemma 8.4

In this section we will prove Lemma 8.4. First we establish the following result.

Lemma 8.6. (*Asymptotics of $f_j(z_k, x)$:*) Let r be a multiple of p and assume the ordering (2.18). The functions $f_j(z_k(x), x)$ in (8.8) behave for $x \rightarrow \infty$ as

$$\begin{aligned} f_0(z_0(x), x) &= (-1)^r x^r + O(x^{r-p-1}), \\ f_0(z_k(x), x) &= O(x^{r-p-1}), \end{aligned} \quad k \in [1 : p], \quad (8.20)$$

and

$$\begin{aligned} f_j(z_0(x), x) &= (-1)^r x^{r-j} + O(x^{r-j-p-1}), \\ f_j(z_k(x), x) &= C_{j,k} x^{r-j} + O(x^{r-j-p-1}), \quad k \in [1 : j], \\ f_j(z_k(x), x) &= O(x^{r-j-p-1}), \quad k \in [j+1 : p], \end{aligned} \quad (8.21)$$

for $j \in [1 : p]$, for certain constants $C_{j,k} \neq 0$, $k \leq j$.

Note that the O -terms jump with powers of x^{-p-1} rather than x^{-1} . This is due to the rotational symmetry under rotations with $\exp(2\pi i/(p+1))$.

Lemma 8.6 implies that for $l \geq k$,

$$\begin{pmatrix} f_0(z_0(x)) & \dots & f_0(z_k(x)) \\ \vdots & & \vdots \\ f_{k-1}(z_0(x)) & \dots & f_{k-1}(z_k(x)) \\ f_l(z_0(x)) & \dots & f_l(z_k(x)) \end{pmatrix} = \text{diag}(x^r, x^{r-1}, \dots, x^{r-k+1}, x^{r-l}) \\ \times \begin{pmatrix} 1 & O(x^{-p-1}) & O(x^{-p-1}) & \dots & O(x^{-p-1}) & O(x^{-p-1}) \\ 1 & C_{1,1} & O(x^{-p-1}) & \dots & O(x^{-p-1}) & O(x^{-p-1}) \\ 1 & C_{2,1} & C_{2,2} & \dots & O(x^{-p-1}) & O(x^{-p-1}) \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & C_{k-1,1} & C_{k-1,2} & \dots & C_{k-1,k-1} & O(x^{-p-1}) \\ 1 & C_{l,1} & C_{l,2} & \dots & C_{l,k-1} & C_{l,k} \end{pmatrix}$$

where each $C_{j,k}$ is a non-zero constant. Therefore,

$$\begin{vmatrix} f_0(z_0(x)) & \dots & f_0(z_k(x)) \\ \vdots & & \vdots \\ f_{k-1}(z_0(x)) & \dots & f_{k-1}(z_k(x)) \\ f_l(z_0(x)) & \dots & f_l(z_k(x)) \end{vmatrix} = Cx^{r+(r-1)+\dots+(r-k+1)+(r-l)}(1 + O(x^{-p-1})), \quad (8.22)$$

for some constant $C \neq 0$. Taking ratios of such determinants, we then get the desired Lemma 8.4.

In the rest of this section we prove (8.20)–(8.21). First of all, the statements involving $z_0(x)$ follow easily from the Widom-type formula (2.25) (applied to the antidiagonal reflected symbol (8.14)) taking into account that $z_0(x) \sim x^{-r}$ and $z_1(x), \dots, z_p(x) = O(x^{r/p})$ for $x \rightarrow \infty$, and that $z_0(x) \dots z_p(x) = (-1)^{r+p}/f_p$.

Next, we prove (8.20)–(8.21) for the functions $z_k(x)$ with $k \geq 1$. Let $\mathbf{e}_j \in \mathbb{C}^r$ be the standard basis vector which has all its entries equal to zero except for the entry in position j , which is equal to 1. Let P be the permutation matrix of size $r \times r$ which acts on the vectors \mathbf{e}_j by the rule

$$P\mathbf{e}_{ap+b} = \mathbf{e}_{br/p+a},$$

for any $a \in [0 : r/p - 1]$ and $b \in [0 : p - 1]$. Let D be the $r \times r$ diagonal matrix

$$D := \text{diag}\left(I_p, z^{\frac{p}{r}}I_p, z^{\frac{2p}{r}}I_p, \dots, z^{\frac{r-p}{r}}I_p\right). \quad (8.23)$$

We conjugate the block Toeplitz symbol $F(z, x)$ by the matrices D and P . This results in the following matrix:

$$PDF(z, x)D^{-1}P^{-1} = \begin{pmatrix} A_0 & I & 0 & \dots & 0 & 0 \\ 0 & A_1 & I & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & A_{p-3} & I & 0 \\ 0 & 0 & 0 & 0 & A_{p-2} & I \\ Z & 0 & 0 & 0 & 0 & A_{p-1} \end{pmatrix}, \quad (8.24)$$

with $Z := z^{-\frac{p}{r}} \begin{pmatrix} 0 & I_{r/p-1} \\ 1 & 0 \end{pmatrix}$, and

$$A_j = \begin{pmatrix} -x & 0 & 0 & \dots & 0 & a_{r-p+j} z^{p/r} \\ a_j z^{p/r} & -x & 0 & \ddots & 0 & 0 \\ 0 & a_{p+j} z^{p/r} & -x & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & a_{r-3p+j} z^{p/r} & -x & 0 \\ 0 & 0 & 0 & 0 & a_{r-2p+j} z^{p/r} & -x \end{pmatrix}, \quad (8.25)$$

for $j \in [0 : p-1]$. Note that each of the blocks in (8.24) is a square matrix of size r/p by r/p .

Fix $k \in [1 : p]$. We already know that $z_k(x) \sim C_k x^{r/p}$ as $x \rightarrow \infty$. Hence $z_k^{p/r}(x) \sim C_k^{p/r} x$. From (8.24)–(8.25) and the fact that $\det F(z_k(x), x) = 0$ we see that (see also (2.22))

$$C_k \in \left\{ \prod_{n=0}^{r/p-1} a_{pn}^{-1}, \prod_{n=0}^{r/p-1} a_{pn+1}^{-1}, \dots, \prod_{n=0}^{r/p-1} a_{pn+(p-1)}^{-1} \right\}.$$

Combining this with (2.18) and (1.15), we obtain that for $x \rightarrow \infty$,

$$\begin{aligned} z_1(x) &= \left(\prod_{n=0}^{r/p-1} a_{pn} \right)^{-1} x^{r/p} (1 + O(x^{-p-1})), \\ &\vdots \\ z_p(x) &= \left(\prod_{n=0}^{r/p-1} a_{pn+p-1} \right)^{-1} x^{r/p} (1 + O(x^{-p-1})). \end{aligned} \quad (8.26)$$

Now we consider $\det F^{j-1,0}(z, x)$, i.e., the determinant obtained by skipping the j th row and the first column of $F(z, x)$, $j \in [1 : p]$. Clearly, this determinant is not influenced by the conjugation with the diagonal matrix D in (8.23), in the sense that $\det F^{j-1,0}(z, x) = \det(DFD^{-1})^{j-1,0}(z, x)$. For a matrix A denote with \tilde{A} the matrix obtained by skipping the first row of A and with \hat{A} the matrix obtained by skipping the first column of A . Then from (8.24) we obtain

$$\det F^{j-1,0}(z, x) = \pm \det \begin{pmatrix} \hat{A}_0 & I & & & & \\ & A_1 & I & & & \\ & & \ddots & \ddots & & \\ & & & \tilde{A}_{j-1} & \tilde{I} & \\ & & & A_j & I & \\ & & & & \ddots & \ddots \\ \hat{Z} & & & & & A_{p-2} & I \\ & & & & & & A_{p-1} \end{pmatrix}.$$

Now by repeated Gaussian elimination with the identity matrices I as pivots, the above determinant can be brought to the form

$$\det F^{j-1,0}(z, x) = \pm \det \begin{pmatrix} \pm \tilde{A}_{j-1} \dots A_1 \hat{A}_0 & \tilde{I} \\ \hat{Z} & \pm A_{p-1} A_{p-2} \dots A_j \end{pmatrix}. \quad (8.27)$$

Fix $k \in [1 : p]$. To obtain the dominant behavior of (8.27) for $z = z_k(x)$ as $x \rightarrow \infty$, we should only use the (1, 1) and the (2, 2) blocks in (8.27). Note that both blocks are square. The determinant of the (2, 2) block can be simply factored as $(\det A_j)(\det A_{j+1}) \dots (\det A_{p-1})$ with

$$\det A_i(z = z_k(x)) = \begin{cases} C_{i,k} x^{r/p} + O(x^{r/p-p-1}), & \text{if } k \neq i+1, \\ O(x^{r/p-p-1}), & \text{otherwise,} \end{cases} \quad (8.28)$$

for some $C_{i,k} \neq 0$, thanks to (8.26). The determinant of the (1, 1) block can be expanded by means of the Cauchy-Binet formula:

$$\det(\tilde{A}_{j-1} \dots A_1 \hat{A}_0) = \sum_{m_1, \dots, m_{j-1}=0}^{r/p-1} (\det A_{j-1}^{m_j, m_{j-1}}) \dots (\det A_1^{m_2, m_1}) (\det A_0^{m_1, m_0}), \quad (8.29)$$

where the sum runs over all $(j-1)$ -tuples of integers (m_1, \dots, m_{j-1}) , each of them ranging between 0 and $r/p-1$, with boundary conditions $m_0 = m_j := 0$. We remind the reader that $A^{i,j}$ denotes the submatrix of A obtained by deleting row i and column j . Clearly,

$$\det A_i^{m_{i+1}, m_i}(z = z_k(x)) = \tilde{C}_{i,k} x^{r/p-1} + O(x^{r/p-p-2}), \quad \tilde{C}_{i,k} \neq 0,$$

for all $i \in [0 : j-1]$. Using this in (8.29) we get

$$\det(\tilde{A}_{j-1} \dots A_1 \hat{A}_0)(z = z_k(x)) = C_{j,k} x^{\frac{jr}{p}-j} (1 + O(x^{-p-1})), \quad (8.30)$$

as $x \rightarrow \infty$, for a new constant $C_{j,k}$. This constant $C_{j,k}$ is nonzero, since cancelation of the leading order terms in the sum in (8.29) cannot occur. This is due to Lemma 8.7 below.

By combining (8.27), (8.28) and (8.30), we obtain the desired asymptotics in (8.21) for $z = z_k(x)$ with $k \in [1 : p]$.

The asymptotics in (8.20) can be proved by a similar argument. We now have the relation $z^{-1} \det F^{r-1,0}(z, x) = z^{-p/r} \det(DFD^{-1})^{r-1,0}(z, x)$. For a matrix A denote now with \tilde{A} the matrix obtained by skipping the last (rather than the first) row of A and denote again with \hat{A} the matrix obtained by skipping the first column of A . Then by Gaussian elimination we get the following analogue of (8.27):

$$z^{-1} \det F^{r-1,0}(z, x) = \pm z^{-p/r} \det(z^{-p/r} I \pm \tilde{A}_{p-1} \dots A_1 \hat{A}_0),$$

where $z^{-p/r} I$ arises as the submatrix obtained by skipping the last row and the first column of Z . Clearly, the dominant behavior for $z = z_k(x)$ as $x \rightarrow \infty$ comes from $\pm z^{-p/r} \det(\tilde{A}_{p-1} \dots A_1 \hat{A}_0)$. This determinant can be evaluated using Cauchy-Binet in the same way as before. Then we easily get the asymptotics in (8.20) for $z = z_k(x)$ with $k \in [1 : p]$. This ends the proof of Lemma 8.6. \square

To conclude this section, we state the following lemma which was used above.

Lemma 8.7. *Let A be an $n \times n$ matrix of the form*

$$A = \begin{pmatrix} -b_0 & & & & a_{n-1} \\ a_0 & -b_1 & & & \\ & a_1 & \ddots & & \\ & & \ddots & -b_{n-2} & \\ & & & a_{n-2} & -b_{n-1} \end{pmatrix},$$

with $a_k, b_k > 0$ for all $k \in [0 : n - 1]$. Denote with $A^{k,l}$ the submatrix obtained by skipping the k th row and the l th column of A . Then

$$(-1)^{n+k+l+1} \det A^{k,l} > 0.$$

Proof. Straightforward verification. □

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